# DVfVW Annual Meeting 2016:
## The Modern Tontine: An Innovative Instrument for Longevity Risk Management in an Aging Society

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# Motivation

**Challenge: Ageing Society**

- PAYG-Retirement System under pressure
- Increasing Old-age Demand

**Demands:**

- meet the concerns of an increasing liquidity need at old ages
- be able to provide reliable insurance performance

**Out Solution: Tontine**

- Increasing payout structure
- risk sharing among policyholder
- but volatile payout structure
- large payments only at very high ages

# Tontine Model (1)

- established in $t = 1$, size $N$ fixed
- $W_i$: Total retirement wealth
- $B_i \leq W_i$: one-time tontine investment at entry, no withdrawal
- $q_{i,t}$: one year death probability of $i$ in $t$
- new entrants drawn from corresponding population pyramid
- expected return conditioned that $i$ survives

$$E\left[r_{i,t}|A_{i,t}^{\mathbf{c}}\right] = q_{i,t}B_i$$

- realized return conditioned that $i$ survives

$$r_{i,t}|A_{i,t}^{\mathbf{c}} \approx q_{i,t}B_i \frac{\sum_{\substack{k=1 \\ k\neq i}}^{N} B_k \mathbf{1}_{\{A_{k,t}\}}}{\sum_{\substack{k=1 \\ k\neq i}}^{N} B_k q_{k,t}}$$

# Tontine Model (2)

- If $N \to \infty$

$$Var\left[r_{i,t}|A^{c}_{i,t}\right] = q^2_{i,t}B^2_i - q^2_{i,t}B^2_i = 0$$

- BUT: for fixed $N$, tontine payouts still volatile and follow a Poisson binomial distribution
- Normal approximation for large $N$: $r_{i,t}|A^{c}_{i,t} \sim \mathcal{N}\left(\mu_{i,t}, \sigma_{i,t}\right)$

$$\mu_{i,t} = q_{i,t}B_i$$

$$\sigma_{i,t} = \frac{q_{i,t}Bi}{\sqrt{M-1}}\sqrt{\sum_{m=1}^{M}\left(\frac{\sum_{\substack{k=1\\k\neq i}}^{N} B_k \mathbf{1}^{m}_{\{A_{k,t}\}}}{\sum_{\substack{k=1\\k\neq i}}^{N} B_k q_{k,t}} - 1\right)^2}.$$

# Tontine vs. Annuity

- standard annuity with investment $W_i - B_i$

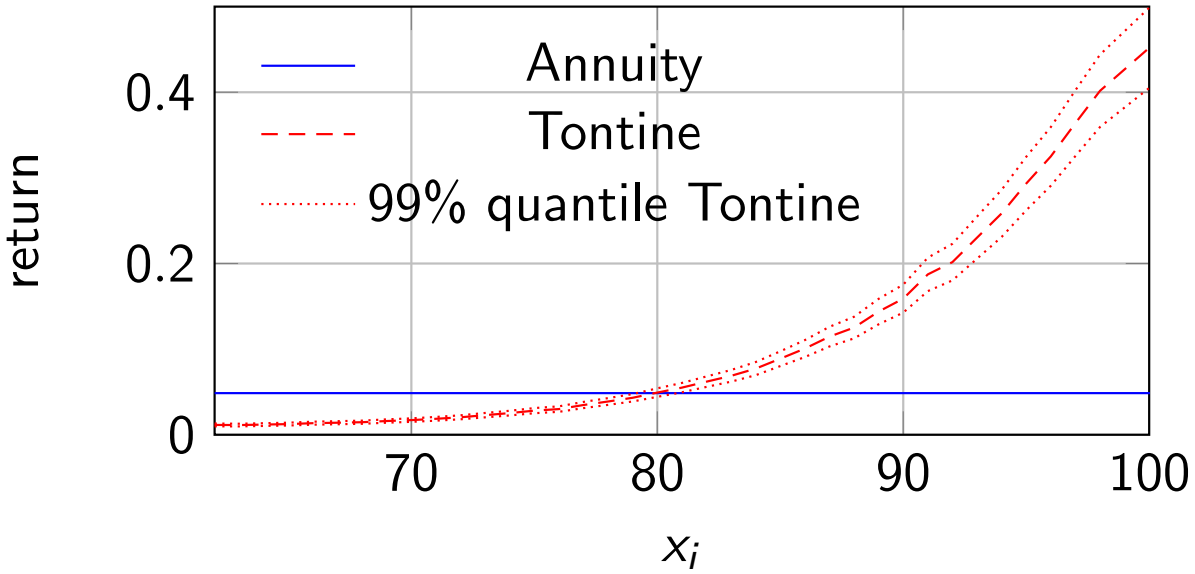



Figure: Normalized Payout of Annuity vs. Tontine

# Liquidity Need Function

- incorporating care costs and medical expenses to consumption → retirement smile

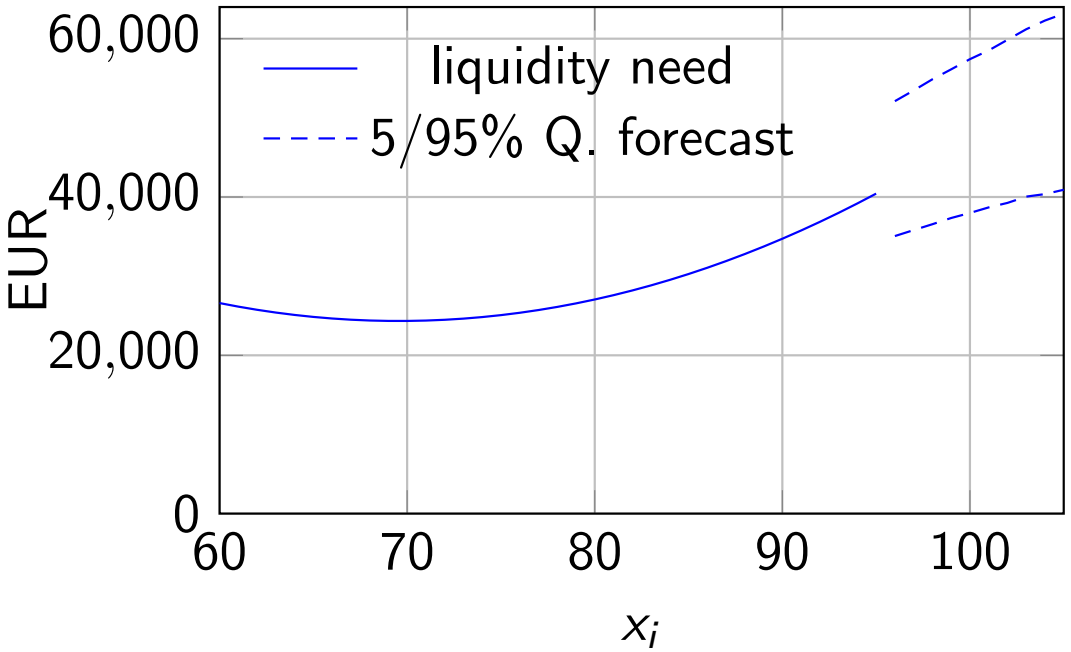



Figure: Aggregated liquidity need per year

$$D_t = \beta_0 + \beta_2 x_t^2 + \beta_1 x_t + \epsilon_t$$

# Multi Cumulative Prospect Theory (1)

$$MCPT\left(i\right) = \sum_{\tau}^{T-x} \delta^{\tau} CPT\left(Z_{i,t+\tau}\right)$$

$$CPT\left(Z_{i,t+\tau}\right) = \int_{-\infty}^{0^-} v\left(z\right) d\left(w^-\left(F_{i,t+\tau}\left(z\right)\right)\right) + \int_{0^+}^{\infty} v\left(z\right) d\left(-w^+\left(1 - F_{i,t+\tau}\left(z\right)\right)\right)$$

- value function $v\left(z\right)$

$$v\left(z\right) = \begin{cases} z^a & z \geq 0 \\ -\lambda \left|z\right|^b & z < 0 \end{cases}$$

$a, b \in (0, 1)$ and $\lambda > 1$

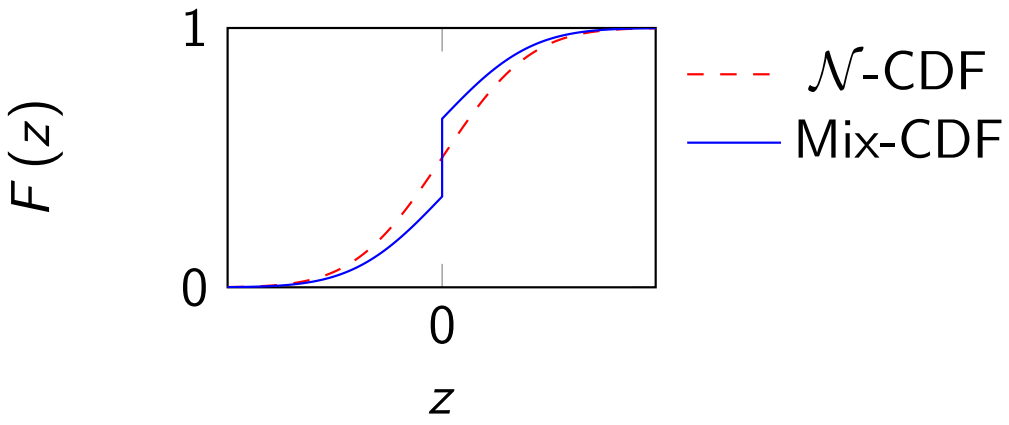
# Multi Cumulative Prospect Theory (2)

- probability weighting function:

$$w^{+}(F) = \frac{F^{\gamma}}{\left(F^{\gamma} + (1-F)^{\gamma}\right)^{1/\gamma}}, \quad w^{-}(F) = \frac{F^{\kappa}}{\left(F^{\kappa} + (1-F)^{\kappa}\right)^{1/\kappa}}$$

- mixture distribution

$$F_{i,t+\tau}(z) = (1 -_{\tau} p_x)\,\mathbf{1}_{[0,\infty)} +_{\tau} p_x \int_{-\infty}^{z} dF_{\mathcal{N}_{i,t+\tau}}(u)$$



# Multi Cumulative Prospect Theory (3)

normally distributed gains/losses with

$$\mu_{i,t} = q_{i,t} B_i + \frac{W_i - B_i}{a_{i,0}} - D_{i,t}$$

and

$$\sigma_{i,t} = \frac{q_{i,t} Bi}{\sqrt{M-1}} \sqrt{\sum_{m=1}^{M} \left( \frac{\sum_{\substack{k=1 \\ k \neq i}}^{N} B_k \mathbf{1}^m_{\{A_{k,t}\}}}{\sum_{\substack{k=1 \\ k \neq i}}^{N} B_k q_{k,t}} - 1 \right)^2}$$

$$\max_{B_i} MCPT(i)$$

$$\text{s. t. } B_i \leq W_i, B_i \geq 0$$

# Calibration Base Case

| Parameter | Notation | Value |
|---|---|---|
| Forecast horizon (in years) | $T$ | 100 |
| Maximum attainable age | $\Omega$ | 105 |
| Fraction of female newborns | $\omega^f$ | 48.68 % |
| Fraction of male newborns | $\omega^m$ | 51.32 % |
| Lower boundary age at tontine entrance | $\underline{x}$ | 63 |
| Upper boundary age at tontine entrance | $\overline{x}$ | 100 |
| Size of the tontine | $N$ | 10,000 |
| Monte Carlo Paths | $M$ | 10,000 |
| Subjective discount factor | $\delta$ | 1 |
| CPT value function parameters | $a, b$ | 0.88 |
| CPT loss sensitivity factor | $\lambda$ | 2.25 |
| CPT $w^+$ parameter | $\gamma$ | 0.61 |
| CPT $w^-$ parameter | $\kappa$ | 0.69 |

Table: MCPT parameters base case

# Results - Base Case (1)

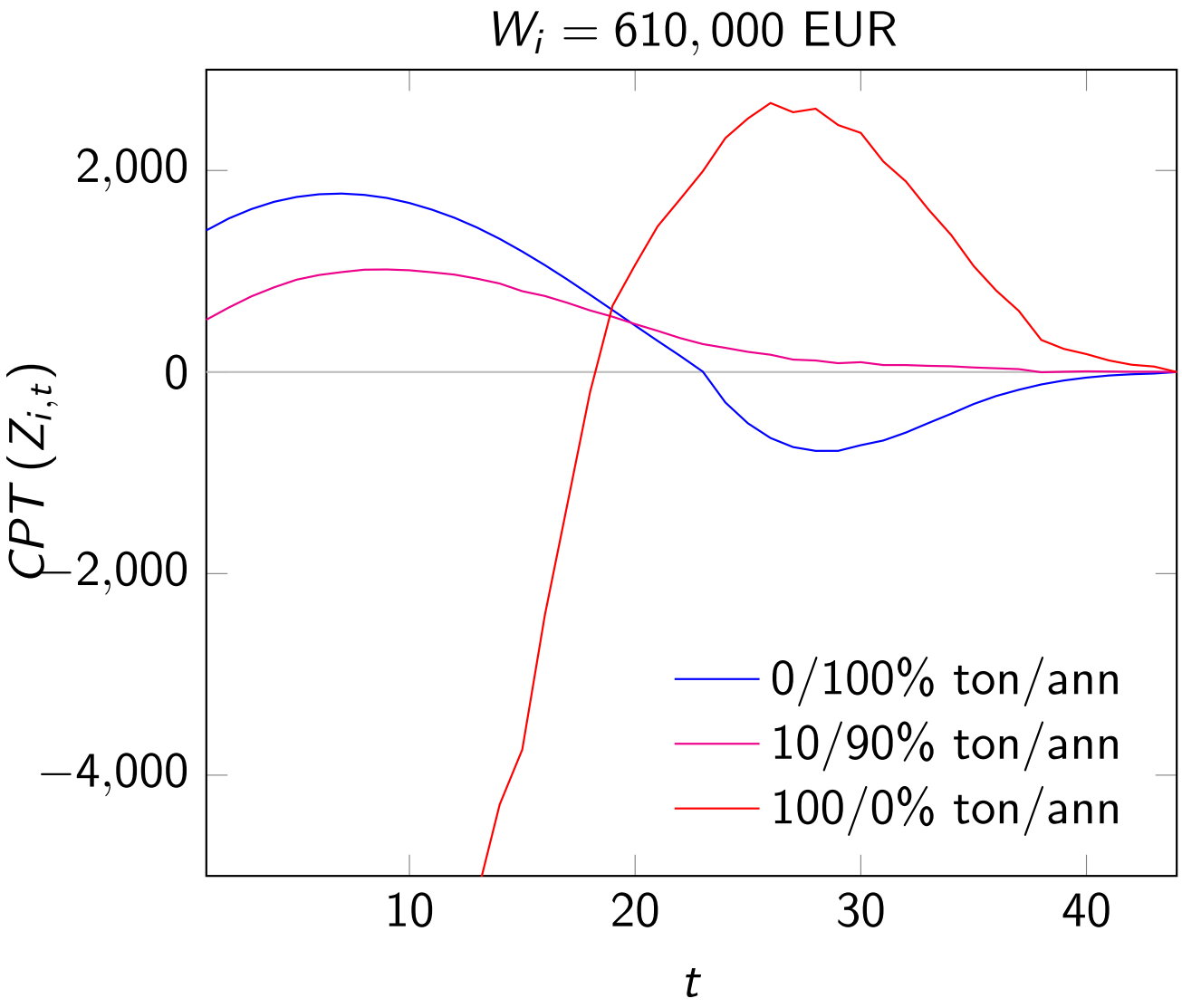



Figure: CPT for different asset allocation

# Results - Base Case (2)

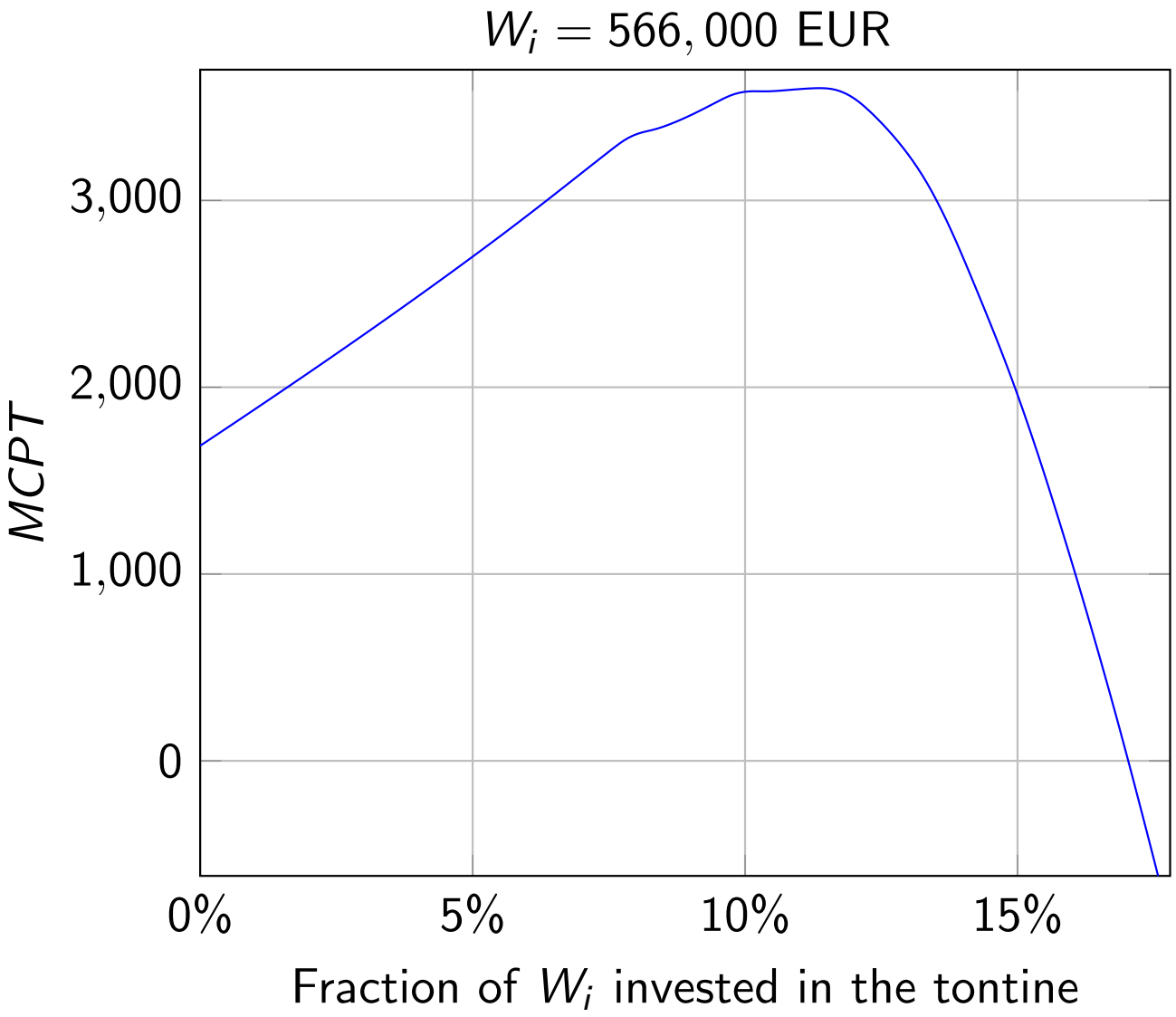



Figure: MCPT

# Results - Base Case (3)

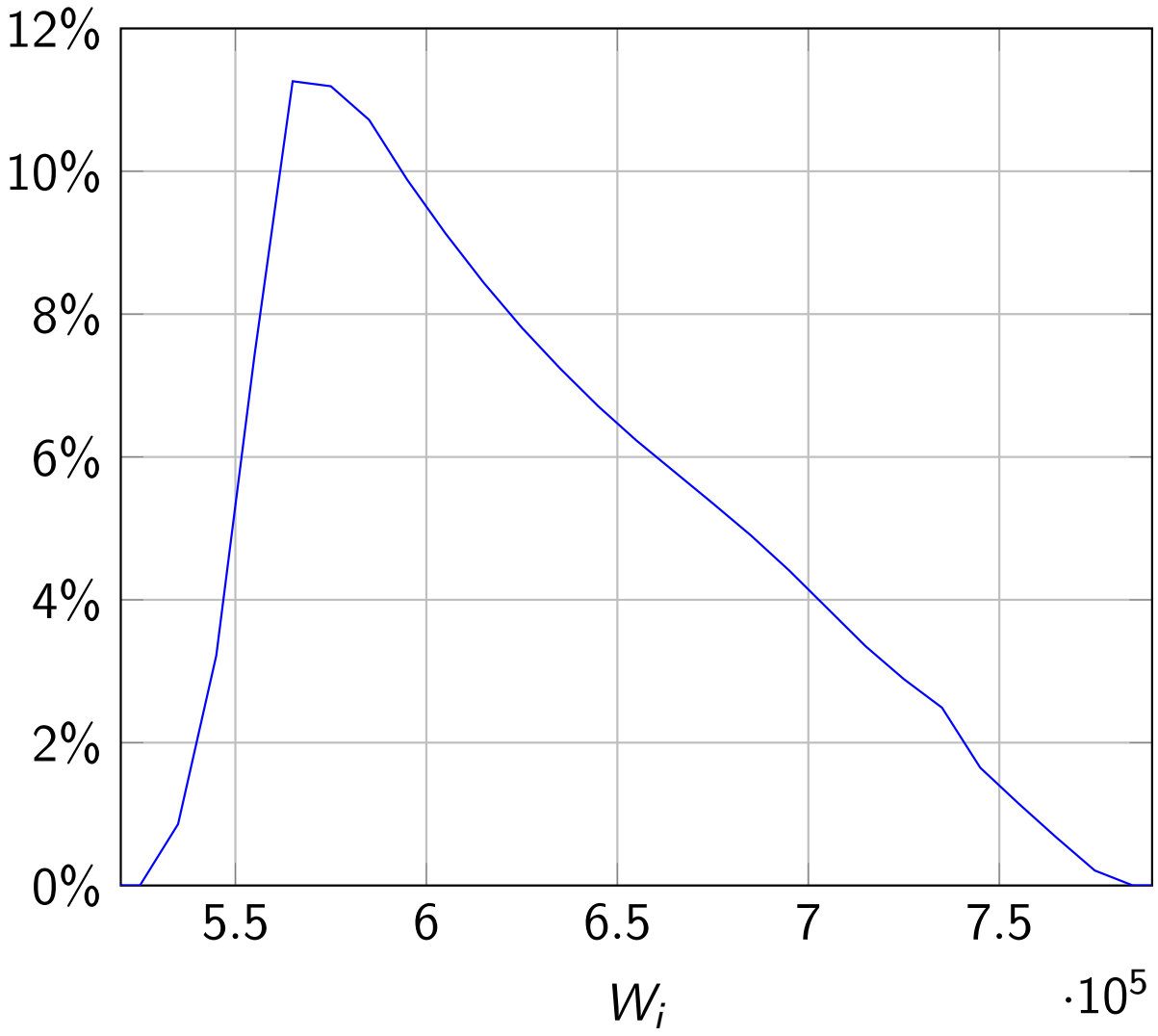



Figure: MCPT utility maximizing fractions to invest in the tontine for different level of $W_i$

# Results - Expected Utility Theory (1)

- $a = b = 0,5, \lambda = 1 \rightarrow$ square root utility
- $\gamma = \kappa = 1$ actual probabilities

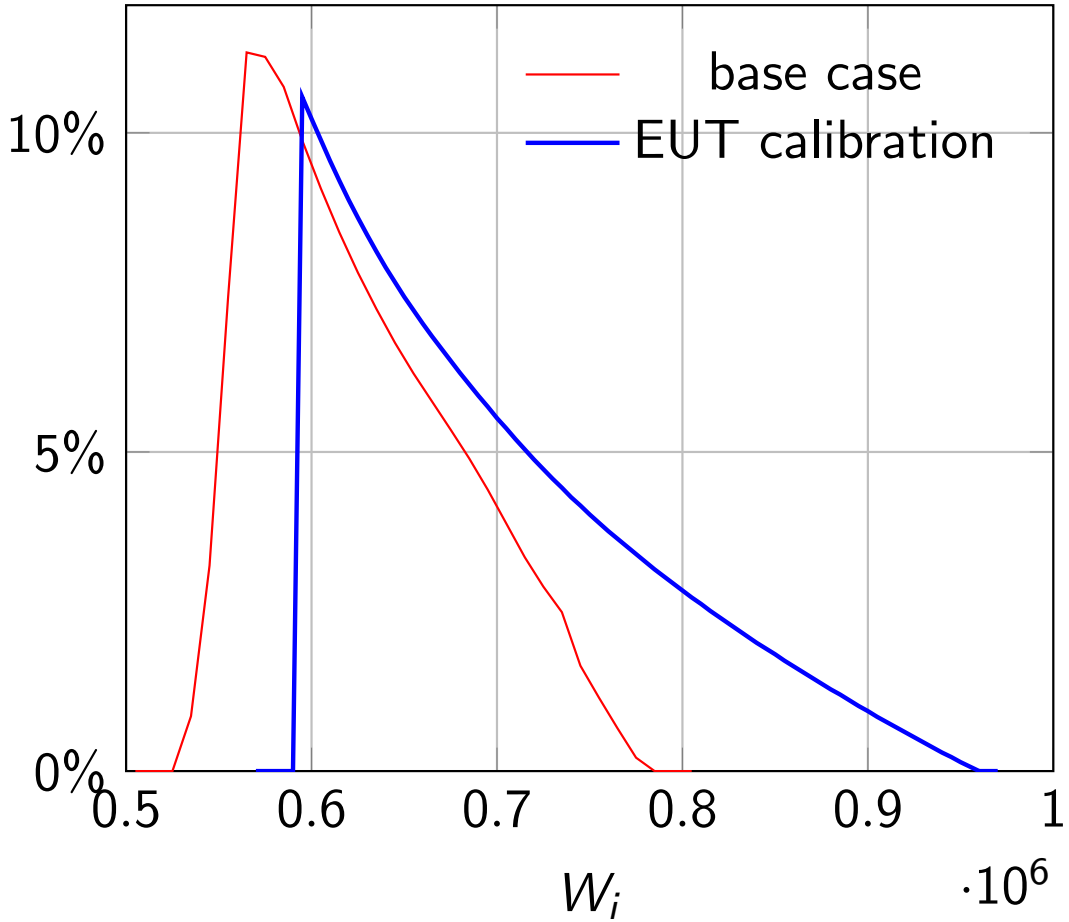
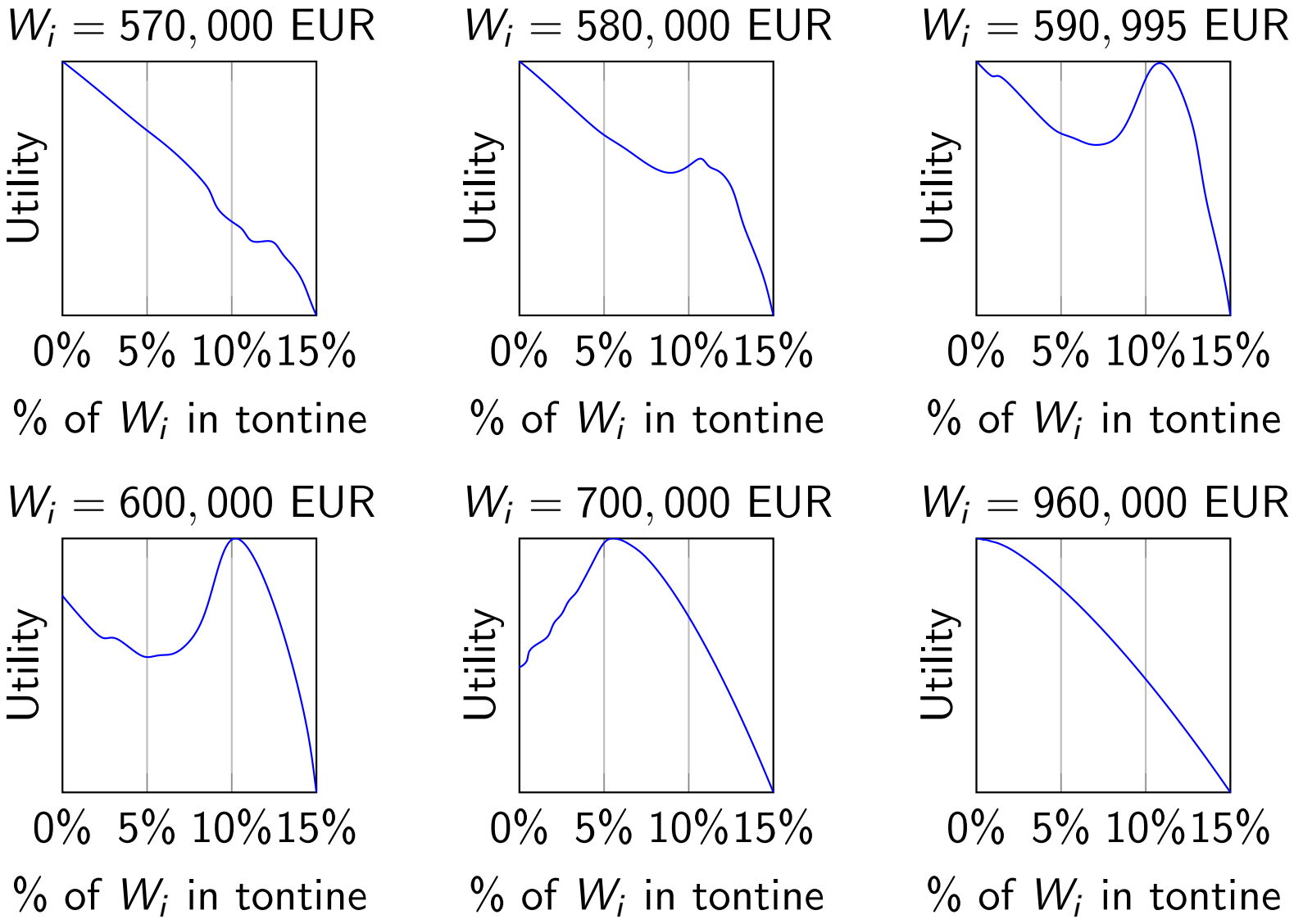



Figure: EUT calibration utility maximizing fractions to invest in the tontine for different level of $W_i$

# Results - Expected Utility Theory (2)



# Further Results

- **increase in tontine size by** $\alpha \rightarrow$ volatility declines by $\sqrt{\alpha}$
- **volatile liquidity need** $\rightarrow$ investment pattern shifts to the right and less tontine investment optimal
- **subjective mortality (improvement)** $\rightarrow$ optimal to begin investing more for lower and higher $W_i$
- **changing liquidity need** $\rightarrow$ parallel upward shift: investment pattern shifts to higher $W_i$, lower peak. exponential shape: generally higher tontine level optimal, peak at 20%

# Backup

- Probability that $i$ dies in $t$:

$$P\left(A_{i,t}\right) = q_{i,t}$$

- Probability that $i$ survives in $t$:

$$P\left(A_{i,t}^{c}\right) = 1 - q_{i,t}$$

- Probability that at least someone dies in $t$ is

$$P\left(A_{0,t}^{c}\right) = P\left(\bigcup_{i=1}^{N} A_{i,t}\right)$$

- Probability that no one dies in $t$:

$$P\left(A_{0,t}\right) = 1 - P\left(\bigcup_{i=1}^{N} A_{i,t}\right)$$

- Probability that $k$ dies in $t \mid$ at least someone dies in $t$

$$P\left(A_{k,t} \mid A_{0,t}^{c}\right) = \frac{P\left(A_{k,t}\right)}{P\left(A_{0,t}^{c}\right)} = \frac{q_{k,t}}{P\left(\bigcup_{i=1}^{N} A_{i,t}\right)} = \rho_{k,t}$$

# Backup

- if $k \neq i$ dies:

$$0 \leq a_{i,k,t} \leq 1 \text{ for } i, k = 1 \ldots N \text{ and } i \neq k$$

$$a_{k,k,t} = -1 \text{ for } k = 1 \ldots N$$

$$\sum_{i=1}^{N} a_{i,k,t} = 0 \text{ for } k = 1 \ldots N$$

$$E\left[r_{i,t}\right] = \underbrace{E\left[r_{i,t} \mid A_{0,t}\right]}_{0} P\left(A_{0,t}\right) + \underbrace{E\left[r_{i,t} \mid A_{0,t}^{c}\right]}_{\sum_{k=1}^{N} \rho_{k,t} a_{i,k,t} B_k} P\left(A_{0,t}^{c}\right) \stackrel{!}{=} 0$$

- solve for the $a_{i,k,t}$'s that yield $E\left[r_{i,t} \mid A_{0,t}^{c}\right] = 0$

# Backup

- for large $N$, $a_{i,k,t}$'s approximately constant

$$\overline{a}_{i,k,t} \approx \frac{q_{i,t} B_i}{\sum\limits_{\substack{k=1 \\ k \neq i}}^{N} q_{k,t} B_k}$$

- realized return conditioned that $i$ survives becomes

$$r_{i,t} | A_{i,t}^{\mathbf{c}} \approx q_{i,t} B_i \frac{\sum\limits_{\substack{k=1 \\ k \neq i}}^{N} B_k \mathbf{1}_{\{A_{k,t}\}}}{\sum\limits_{\substack{k=1 \\ k \neq i}}^{N} B_k q_{k,t}}$$

- If $N \to \infty$, $E\left[\mathbf{1}_{\{A_{k,t}\}}\right] \to q_{k,t}$

$$Var\left[r_{i,t} | A_{i,t}^{\mathrm{c}}\right] = q_{i,t}^2 B_i^2 - q_{i,t}^2 B_i^2 = 0$$

# Backup

- expected return conditioned that $i$ survives in $t$

$$E\left[r_{i,t}|A_{i,t}^{\mathbf{c}}\right] = q_{i,t}B_i$$

- unconditional realized benefit for $i$ in $t$:

$$r_{i,t} = \sum_{\substack{k=1\\k\neq i}}^{N} a_{i,k,t}B_k\mathbf{1}_{\left\{A_{k,t}\bigcap A_{i,t}^{c}\right\}} - B_i\mathbf{1}_{\{A_{i,t}\}}$$

- realized return conditioned that $i$ survives:

$$r_{i,t}|A_{i,t}^{\mathbf{c}} = \sum_{\substack{k=1\\k\neq i}}^{N} a_{i,k,t}B_k\mathbf{1}_{\{A_{k,t}\}}$$