Herbsttagung DVfVW / DGVFM 2015

Risikomanagement –
Modelle, Risikomaße, Abhängigkeiten

Copulas als Hilfsmittel zur Modellierung von Abhängigkeiten in der Versicherung

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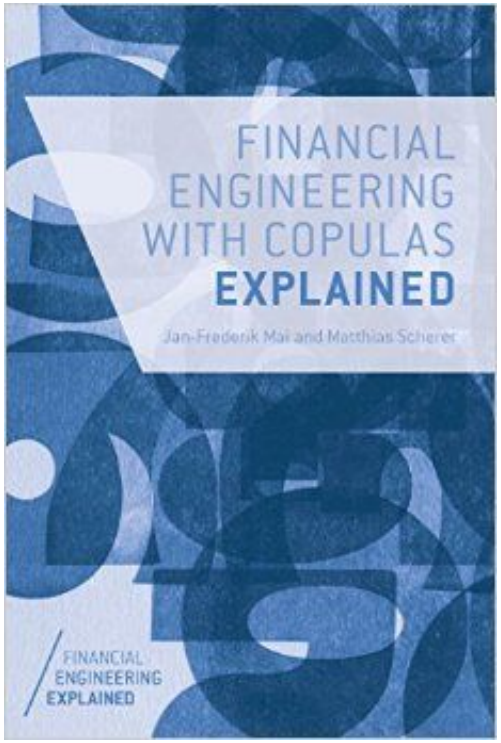


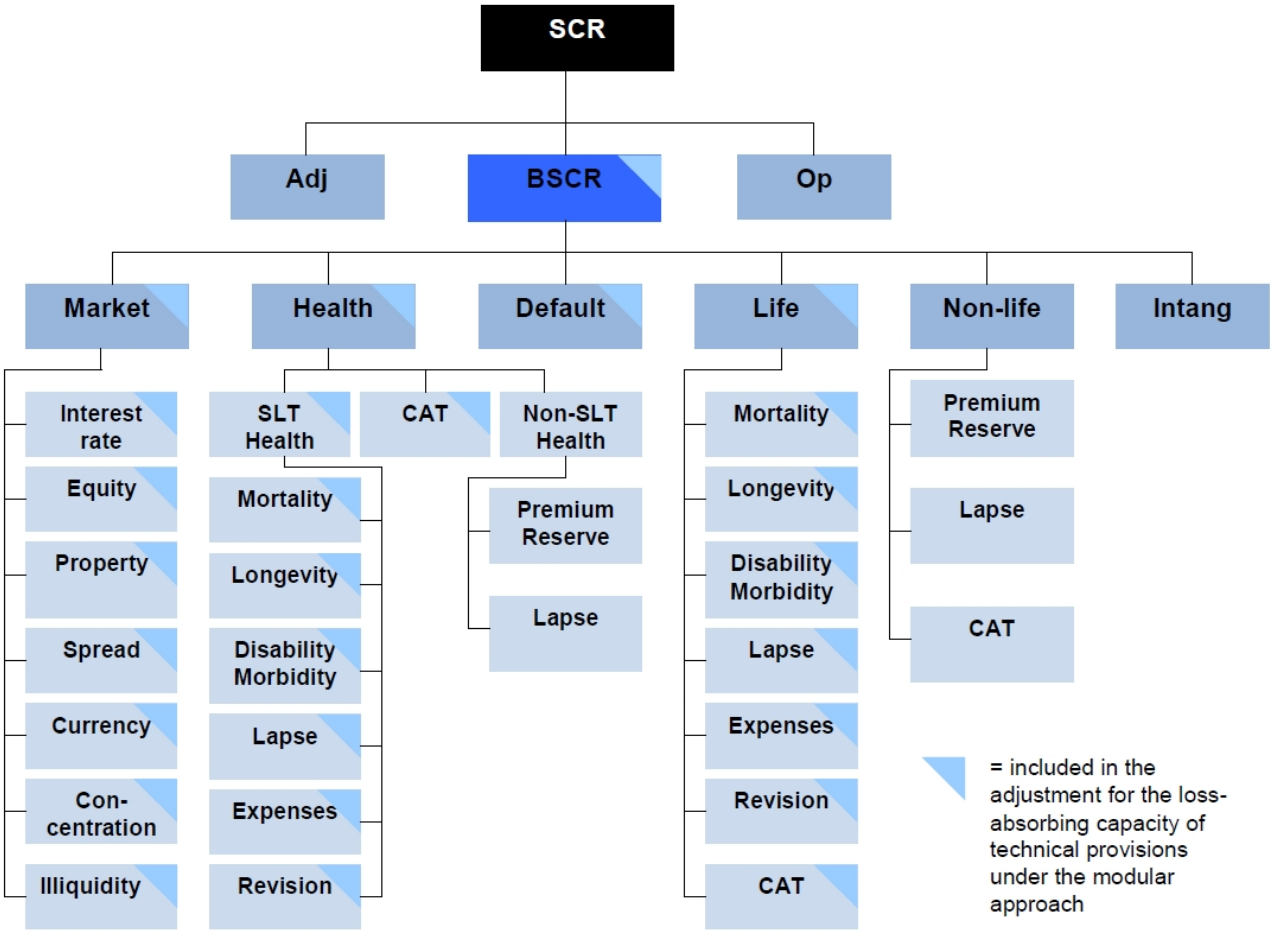
Motivation: Insurance mathematics “anno domini”

Cramér–Lundberg model

$$X_t = \underbrace{x}_{\text{initial surplus}} + \underbrace{c\,t}_{\text{incoming premium}} - \underbrace{\sum_{i=1}^{N_t} \xi_i}_{\text{outgoing claims}}$$

- **i.i.d.** claim sizes $\{\xi_i\}_{i\in\mathbb{N}}$
- **i.i.d.** inter arrival times (exponentially distributed)

[De?]Motivation: Solvency II



Source: QIS5 Technical Specifications, page 90
http://archive.eiopa.europa.eu/fileadmin/tx_dam/files/consultations/QIS/QIS5/QIS5-technical_specifications_20100706.pdf

Motivation

Modeling challenges for insurance companies

- Risk aggregation
- Operational risk
- Market risk (Multivariate asset returns, interest rates, credit spreads, etc.)
- Life testing & mortality (Dependent decrements, coupled lives)
- Reinsurance premium for portfolios (non-linear payoffs)
- Claim counts & sizes
- Insurance pricing
- ...

$\Rightarrow$ In all cases, we work with **dependent random variables**!

Agenda

What can copulas do for you?

1) They **describe and measure dependence** between random variables

2) They make it possible to **identify dependence**

3) They allow us to **construct new multivariate distributions**, with

   - arbitrary marginal laws,

   - all sorts of dependence structures

$\Rightarrow$ Short: **They separate marginal laws from dependence**

Requirements

What do you need to know for these tasks?

A) A toolbox with **different copula families**

B) Understanding the **analytical and statistical properties** of different copulas

C) **Simulation** and **estimation** strategies

D) Understanding of **dependence measures**

⇒ Short: **More than we cover in this introduction**

Definition

$$C : [0,1]^d \to [0,1]$$

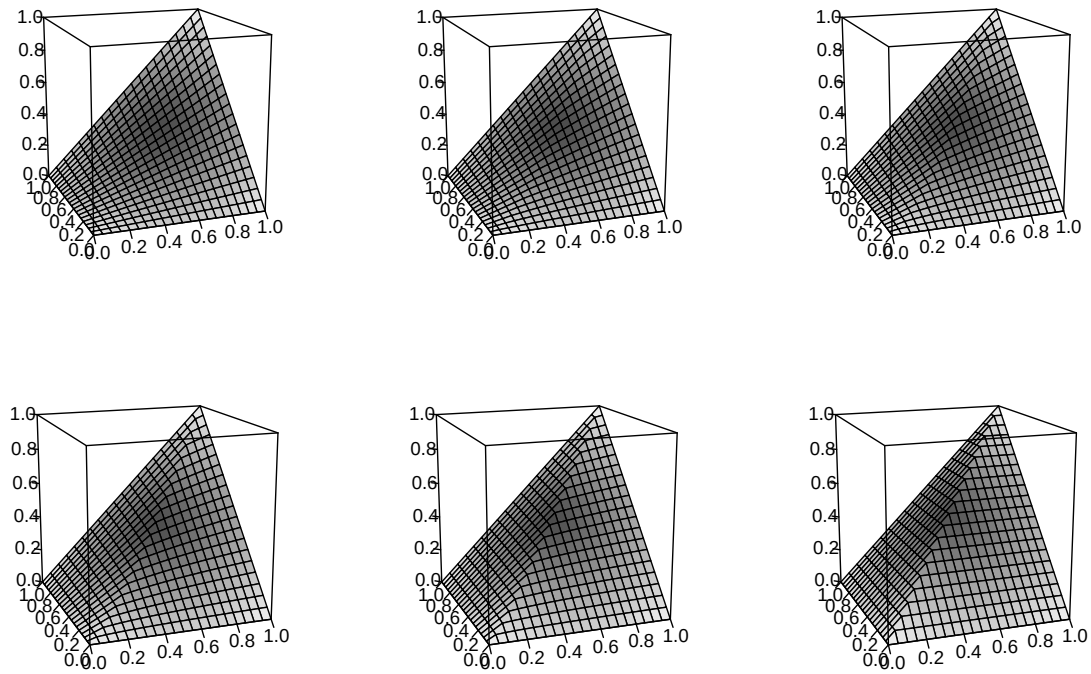
is called **copula**, if there is a random vector $(U_1, \ldots, U_d)$ such that:

i) Each margin U_k is uniform on $[0,1]$, i.e. $U_k \sim \mathcal{U}[0,1]$,

ii) C is the joint distribution of $(U_1, \ldots, U_d)$, i.e.

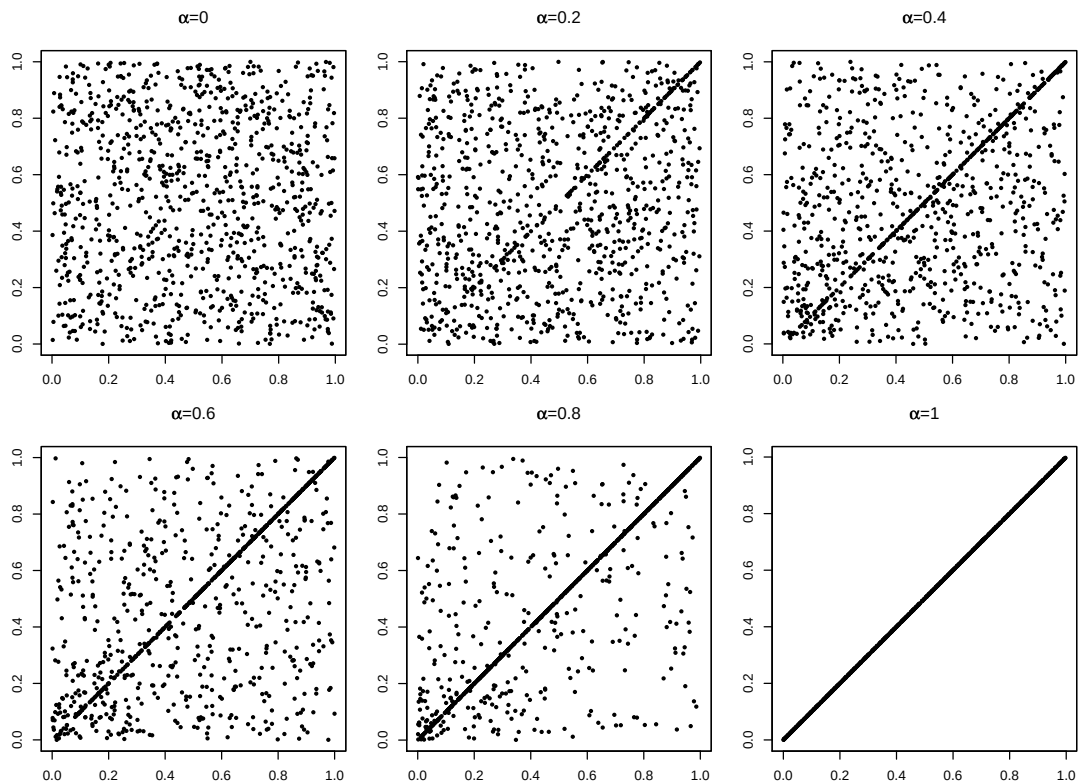
$$C(u_1, \ldots, u_d) = \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d), \quad u_1, \ldots, u_d \in [0,1]$$

Graphical visualization as functions



Example: Cuadras–Augé copula $C_\alpha(u_1, u_2) = \min\{u_1, u_2\} \max\{u_1, u_2\}^{1-\alpha}$, $\alpha = 0$ (upper left), $\alpha = 0.2$, $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1$ (lower right)

Graphical visualization as scatter plots



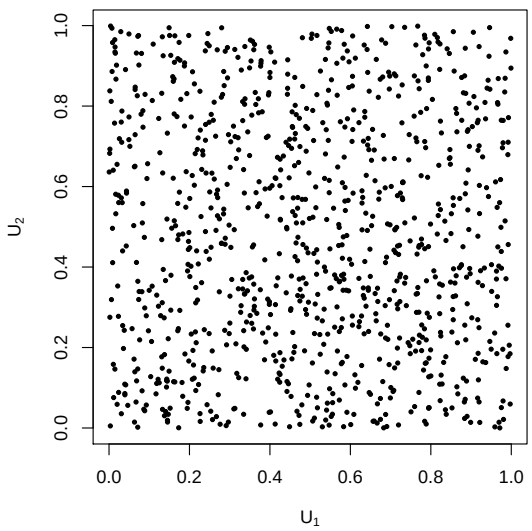
Interpretation: $\dfrac{\text{number of points in } B}{\text{number of all points}} \approx \mathbb{P}((U_1, U_2) \in B) = \mathrm{d}C(B), \quad B \in \mathcal{B}(\mathbb{R}^2)$

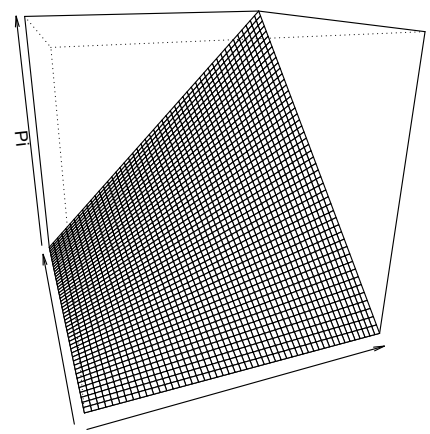
A) First examples

Independence copula

- Consider 2 independent random variables $U_k \sim \mathcal{U}[0,1]$, $k = 1, 2$
- The joint distribution of (U_1, U_2) is the **independence copula**

$$\Pi(u_1, u_2) := u_1\, u_2, \quad u_1, u_2 \in [0,1]$$



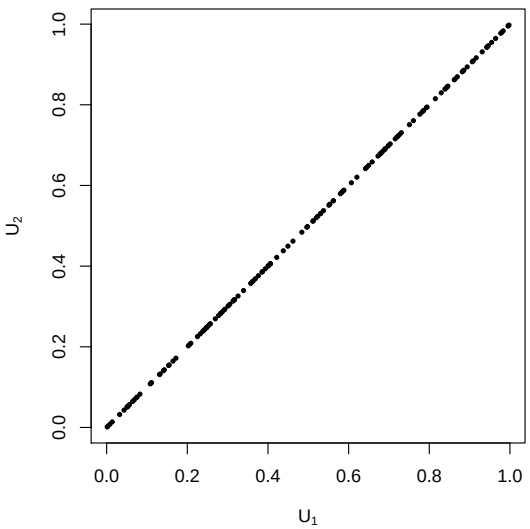


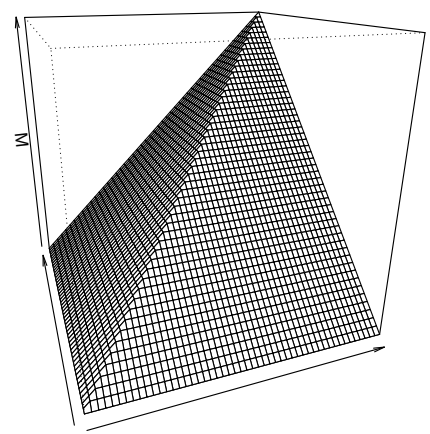
A) First examples

Comonotonicity copula

- Consider for $U \sim \mathcal{U}[0,1]$ the vector (U, U)
- The joint distribution of (U, U) is the **comonotonicity copula**

$$M(u_1, u_2) := \min\{u_1, u_2\}, \quad u_1, u_2 \in [0,1]$$



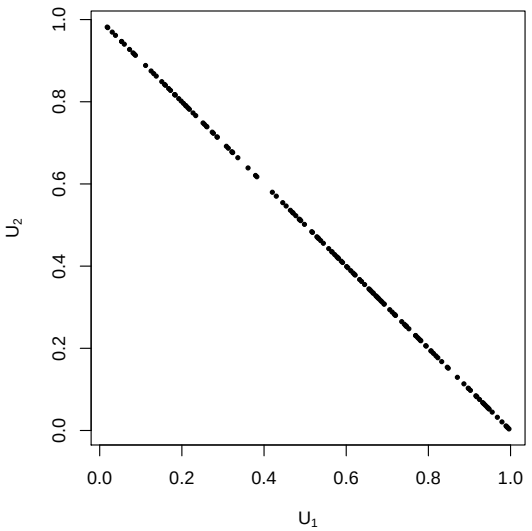


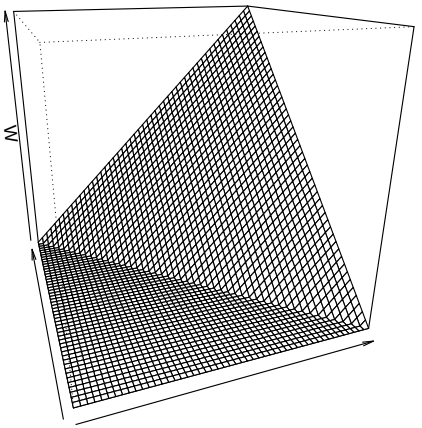
A) First examples

Countermonotonicity copula

- Consider for $U \sim \mathcal{U}[0,1]$ the vector $(U, 1-U)$
- The joint distribution of $(U, 1-U)$ is the **countermonotonicity copula**

$$W(u_1, u_2) := (u_1 + u_2 - 1)\,\mathbf{1}_{\{u_1+u_2 \geq 1\}}, \quad u_1, u_2 \in [0,1]$$





Agenda

What can copulas do for you?

1) They **describe and measure dependence** between random variables

2) They make it possible to **identify dependence**

3) They allow us to **construct new multivariate distributions**

1) Copulas describe dependence

Sklar's Theorem - the fundamental theorem of copula theory

- $F : \mathbb{R}^d \to [0,1]$ is the distribution function of some random vector $(X_1, \ldots, X_d)$ if and only if there exist a copula $C : [0,1]^d \to [0,1]$ and univariate distribution functions $F_1, \ldots, F_d : \mathbb{R} \to [0,1]$ such that

$$C(F_1(x_1), \ldots, F_d(x_d)) = F(x_1, \ldots, x_d), \quad x_1, \ldots, x_d \in \mathbb{R}$$

- The distribution of X_j equals F_j and the correspondence between F and C is one-to-one if all functions $F_1, \ldots, F_d$ are continuous

- Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate cdfs, then

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

  is a valid d-dimensional distribution function

1) Copulas describe dependence

Typical applications of Sklar's Theorem

(1) $\mathbf{F} \rightsquigarrow \mathbf{C} \oplus \mathbf{F_1}, \ldots, \mathbf{F_d}$: **Model analysis**

→ Break difficult analysis of dependent random variables into simpler tasks

→ Estimation of multivariate models

(2) $\mathbf{C} \oplus \mathbf{F_1}, \ldots, \mathbf{F_d} \rightsquigarrow \mathbf{F}$: **Model design**

→ Given $F_1, \ldots, F_d$, impose an arbitrary copula C on them to obtain F

→ For instance, useful for modeling tasks where margins are input

In a nutshell:

Joint distribution F = Copula C $\oplus$ Marginal distributions $F_1, \ldots, F_d$

1) Copulas measure dependence

⚠ Correlation is often misleading

- It only measures linear dependence
- It depends on the marginal laws

CORRELATION AND DEPENDENCY IN RISK MANAGEMENT: PROPERTIES AND PITFALLS

PAUL EMBRECHTS, ALEXANDER MCNEIL, AND DANIEL STRAUMANN

ABSTRACT. Modern risk management calls for an understanding of stochastic dependence going beyond simple linear correlation. This paper deals with the static (non-time-dependent) case and emphasizes the copula representation of dependence for a random vector. Linear correlation is a natural dependence measure for multivariate normally and, more generally, elliptically distributed risks but other dependence concepts like comonotonicity and rank correlation should also be understood by the risk management practitioner. Using counterexamples the falsity of some commonly held views on correlation is demonstrated; in general, these fallacies arise from the naive assumption that dependence properties of the elliptical world also hold in the non-elliptical world. In particular, the problem of finding multivariate models which are consistent with prespecified marginal distributions and correlations is addressed. Pitfalls are highlighted and simulation algorithms avoiding these problems are constructed.

P. Embrechts, A. McNeil, D. Straumann (2002),
Correlation and dependence in risk management: properties and pitfalls

Alternative: Concordance measures

- **Kendall's τ**

$$\tau_C = 4\,\mathbb{E}[C(U_1,U_2)] - 1 = \mathbb{P}(\underbrace{(U_1-V_1)(U_2-V_2)>0}_{\text{concordant pair}}) - \mathbb{P}(\underbrace{(U_1-V_1)(U_2-V_2)<0}_{\text{discordant pair}})$$

- **Spearman's ρ_S**

$$\rho_S = \mathrm{Cor}(F_1(X_1), F_2(X_2)) = 12\int_0^1\int_0^1 (C(u_1,u_2) - u_1\,u_2)\mathrm{d}u_1\,\mathrm{d}u_2$$
$$= 3\Big(\mathbb{P}((U_1-V_1)(U_2-W_2)>0) - \mathbb{P}((U_1-V_1)(U_2-W_2)<0)\Big),$$

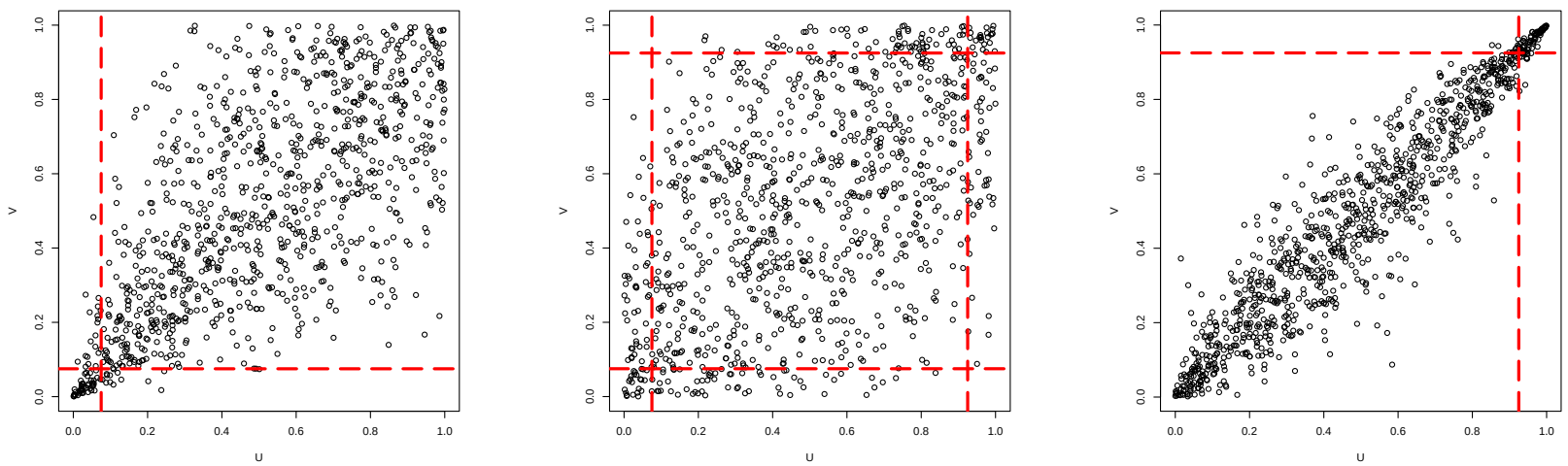
for independent copies (U_1,U_2), (V_1,V_2), and (W_1,W_2) with cdf C

1) Copulas measure dependence

Upper- and lower-tail dependence

- For a bivariate copula C, the coefficients of **upper- and lower-tail dependence** UTD_C and LTD_C are defined as (provided the limit exists)

$$UTD_C = \lim_{u \uparrow 1} \frac{C(u,u) - 2\,u + 1}{1 - u}, \quad LTD_C = \lim_{u \downarrow 0} \frac{C(u,u)}{u}$$



B) Analytical properties of copulas

Fréchet–Hoeffding bounds:

- Let $C : [0,1]^2 \to [0,1]$ be any copula. Then

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2), \quad (u_1, u_2) \in [0,1]^2$$

- **Interpretation**:

  $\to$ Comonotonicity copula M: strongest dependence

  $\to$ Countermonotonicity copula W: strongest negative dependence

B) Analytical properties of copulas

Invariance under strictly $\nearrow$ transformations:

- Let (X_1, X_2) have continuous margins and copula C
- For strictly $\nearrow$ functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, the copula of $(g_1(X_1), g_2(X_2))$ is again C

- This is used to transform / standardize the marginal laws:

  $\rightarrow$ For example: It allows to change scales or currencies without altering C

B) Analytical properties of copulas

Copula density

- Given some technical conditions, there is a function $c : [0,1]^2 \to [0,\infty)$ s.t.

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} c(v_1, v_2)\, \mathrm{d}v_2 \mathrm{d}v_1, \quad u_1, u_2 \in [0,1]$$

- The copula C is then called **absolutely continuous** with **copula density** c
- Then, the density is given by

$$c(u_1, u_2) = \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} C(u_1, u_2)$$

A) More copula families

Archimedean copulas

- Their analytical form is (φ satisfies certain monotonicity properties)

$$C_\varphi(u_1, \ldots, u_d) = \varphi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d))$$

- Many families exist – they differ by choice of φ and their properties
- Kendall's τ can be derived as a functional of φ

$$\tau_{C_\varphi} = 1 + 4 \int_0^1 \frac{\varphi^{-1}(x)}{(\varphi^{-1})'(x)} \mathrm{d}x = 1 - 4 \int_0^\infty u(\varphi'(u))^2 \mathrm{d}u$$

- Upper- and lower-tail dependence can directly be inferred

$$UTD_{C_\varphi} = 2 - 2 \lim_{x \searrow 0} \frac{\varphi'(2\,x)}{\varphi'(x)},$$

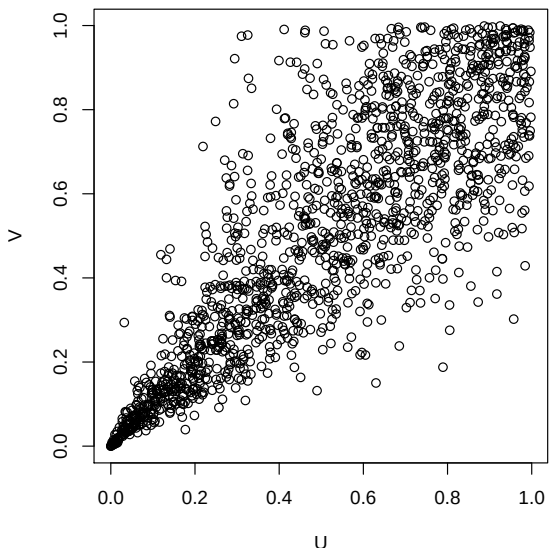
$$LTD_{C_\varphi} = 2 \lim_{x \nearrow \infty} \frac{\varphi'(2\,x)}{\varphi'(x)}$$

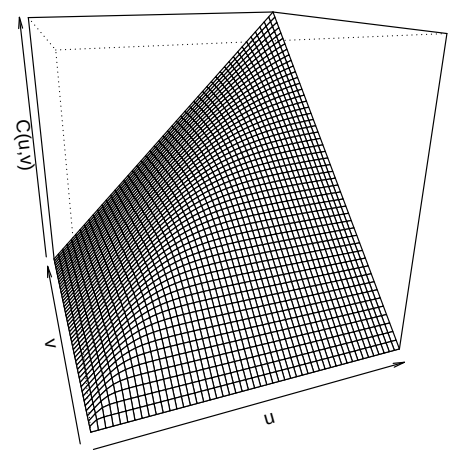
A) More copula families

Archimedean copulas - some examples

- Clayton

$$\varphi(x) = (1+x)^{-1/\vartheta}, \quad \varphi^{-1}(x) = x^{-\vartheta} - 1, \quad \vartheta \in (0, \infty)$$





A) More copula families

Archimedean copulas - some more examples

- AMH

$$\varphi(x) = \frac{1-\vartheta}{e^x - \vartheta}, \quad \varphi^{-1}(x) = \log\Big(\frac{1-\vartheta}{x} + \vartheta\Big), \quad \vartheta \in [0,1)$$

- Frank

$$\varphi(x) = -\frac{1}{\vartheta}\log\left(e^{-x}(e^{-\vartheta}-1)+1\right), \quad \varphi^{-1}(x) = -\log\Big(\frac{e^{-\vartheta x}-1}{e^{-\vartheta}-1}\Big), \quad \vartheta \in (0,\infty)$$

- Gumbel

$$\varphi(x) = e^{-x^{1/\vartheta}}, \quad \varphi^{-1}(x) = \big(-\log(x)\big)^{\vartheta}, \quad \vartheta \in [1,\infty)$$

- Joe

$$\varphi(x) = 1-(1-e^{-x})^{1/\vartheta}, \quad \varphi^{-1}(x) = -\log\left(1-(1-x)^{\vartheta}\right), \quad \vartheta \in [1,\infty)$$

A) More copula families

Elliptical copulas

- An **elliptical copula** is derived from an elliptical distribution F
- Its analytical form is obtained via Sklar's Theorem from the distribution F, i.e.

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0, 1]^d,$$

  where F_k^{-1} are the univariate quantile functions
- Their functional form is not given explicitly
- Two prominent examples: Gaussian and Student-t copula

A) More copula families

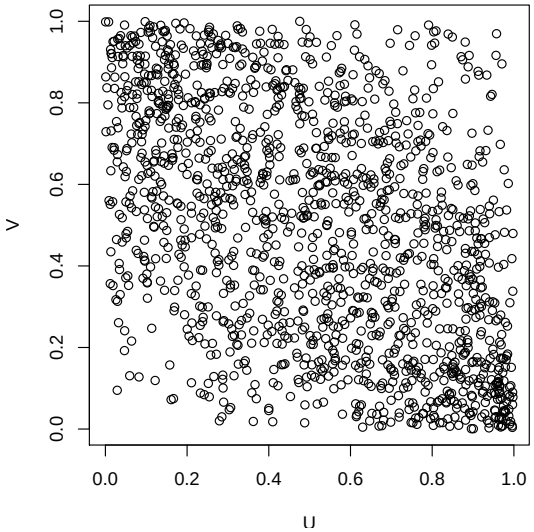
Gaussian copula

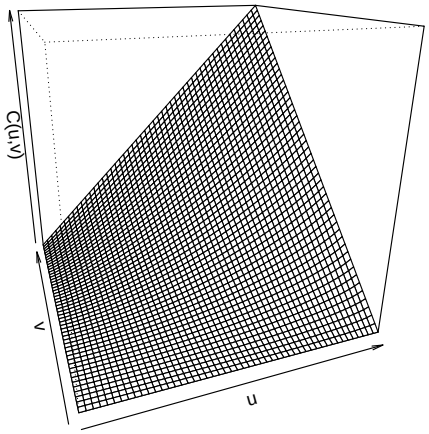
- Given via Sklar's Theorem from the multivariate normal law

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0,1]^d,$$

  where F_k^{-1} are the univariate quantile functions of $\mathcal{N}$, $k = 1, \ldots, d$

- No tail dependence!





A) More copula families

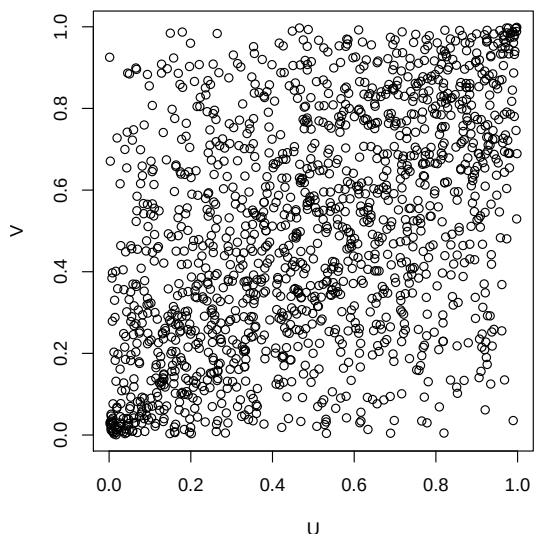
Student t copula

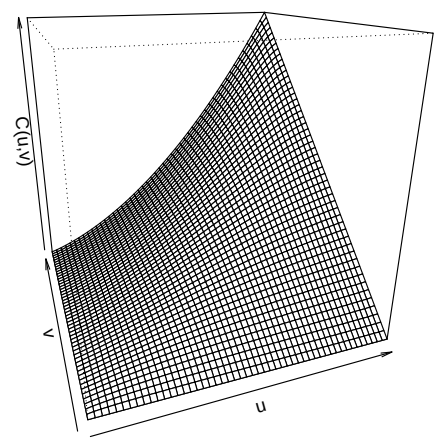
- Given via Sklar's Theorem from the multivariate t law

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0,1]^d,$$

  where F_k^{-1} are the univariate quantile functions of the t distribution

- Positive tail dependence!





Agenda

What can copulas do for you?

1) They **describe and measure dependence** between random variables

2) They make it possible to **identify dependence**

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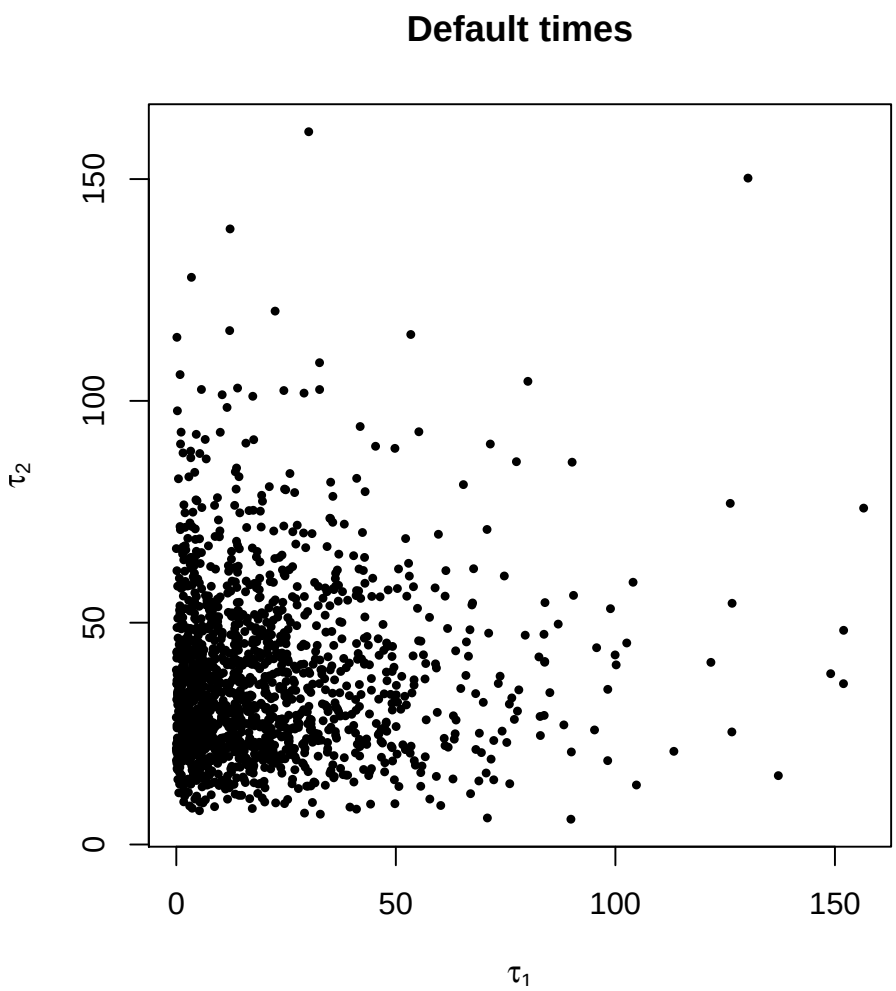
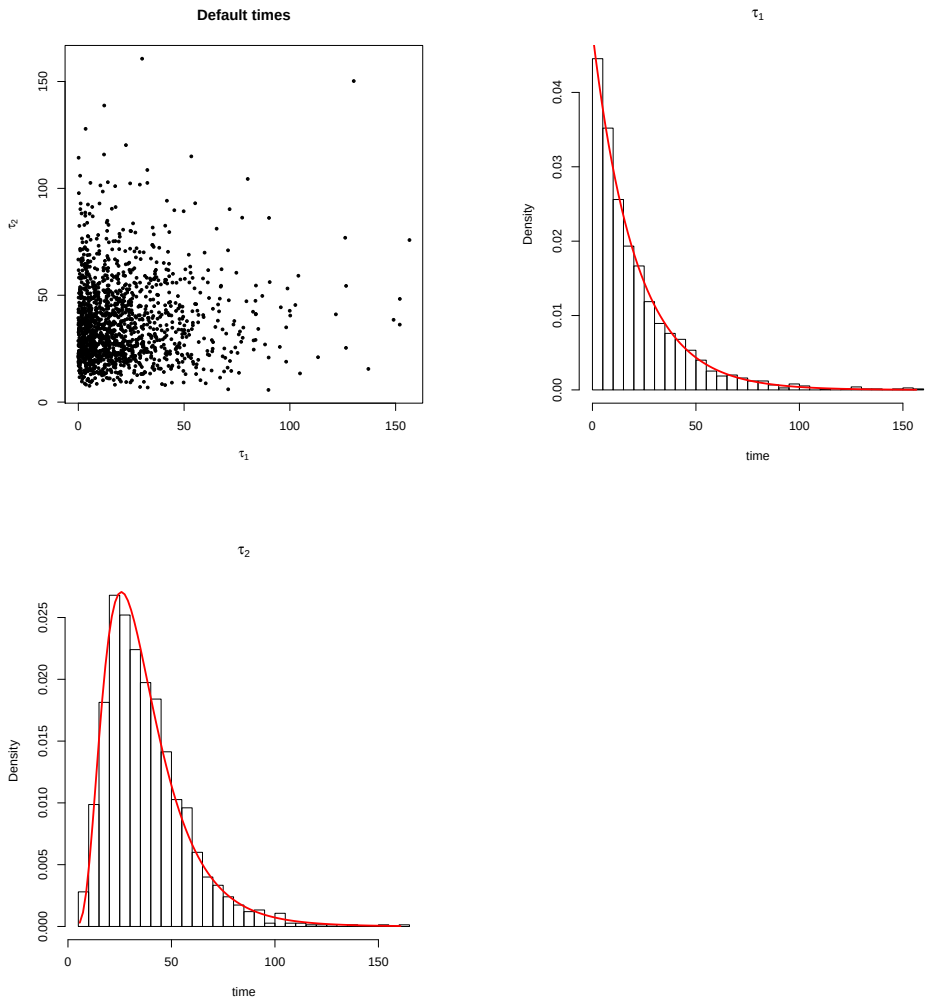
Example: Realizations of two default times (τ_1, τ_2)

Can you guess the dependence?



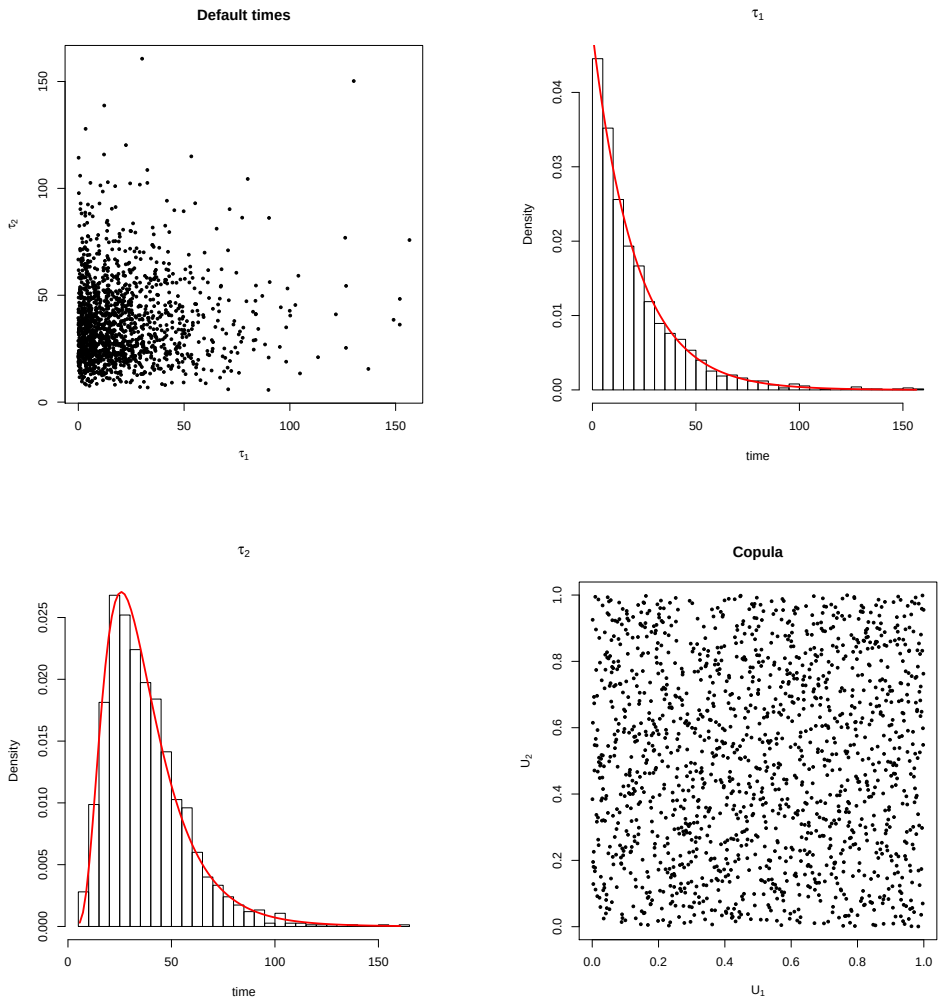
Example: Realizations of two default times (τ_1, τ_2)

Step 1: Identify the marginals



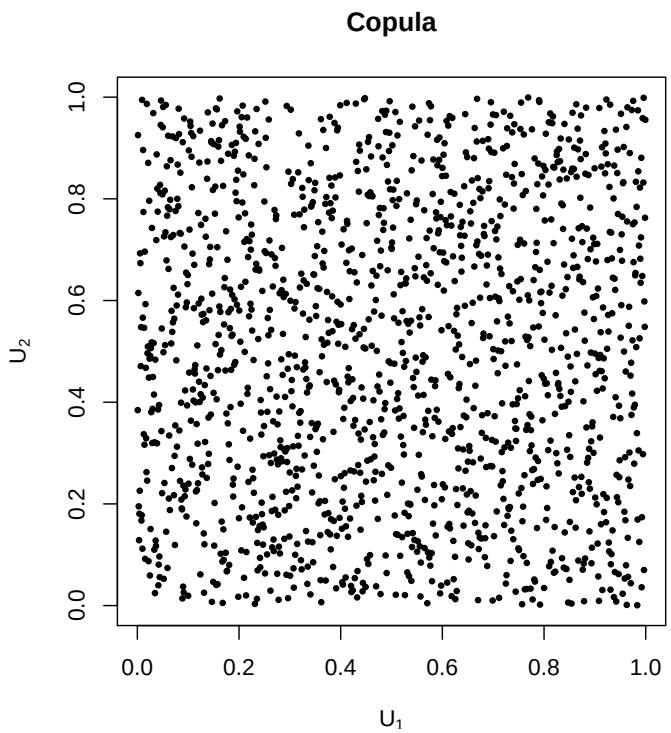
Example: Realizations of two default times (τ_1, τ_2)

Step 2: Transform the marginals to $\mathcal{U}[0,1]$



Example: Realizations of two default times (τ_1, τ_2)

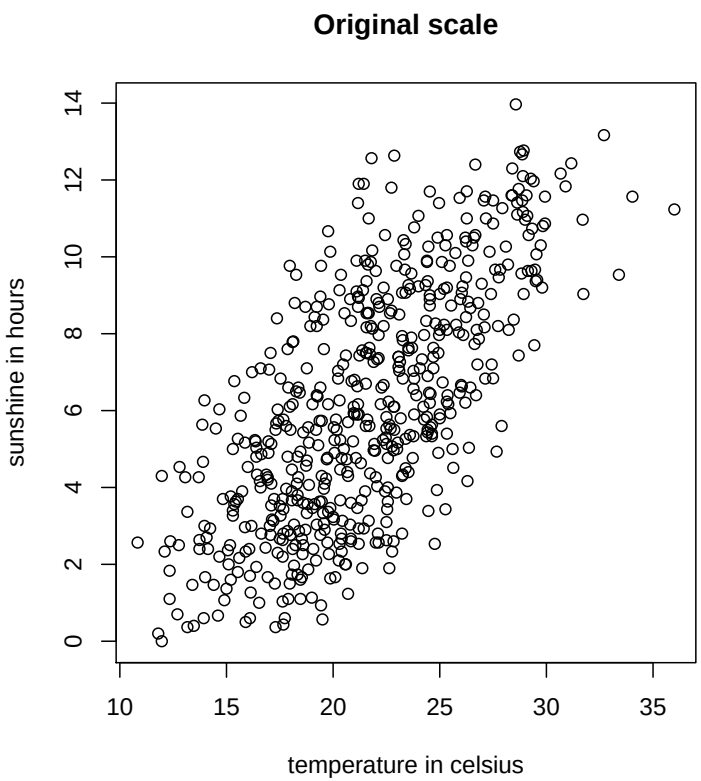
Surprise! It actually was independence



Influence of margins makes it difficult to identify the dependence!

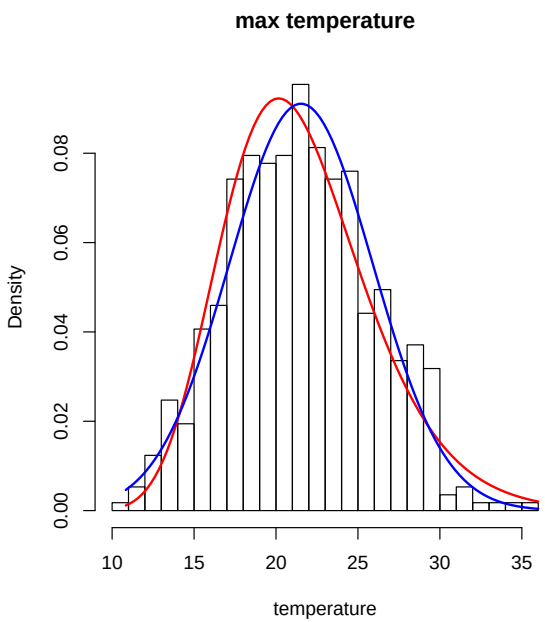
Example: The “Biergarten” weather derivative

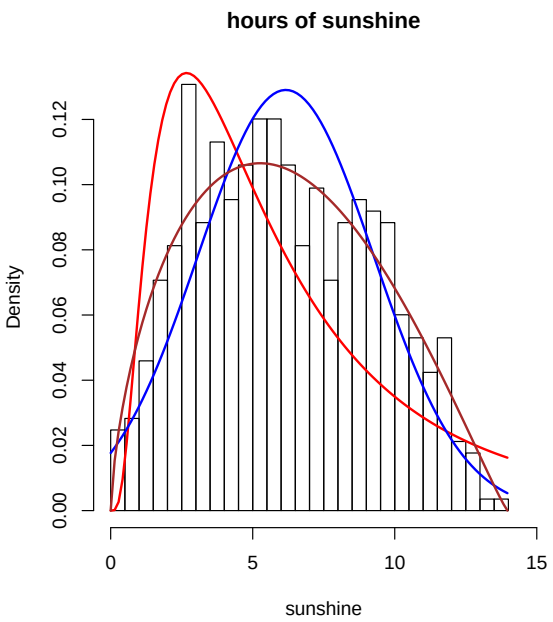
Maximal daily temperature 🌡 and sun duration ☀



(🌡, ☀) of all weekends in August / September since 1948 in Regensburg

Example: The “Biergarten” weather derivative



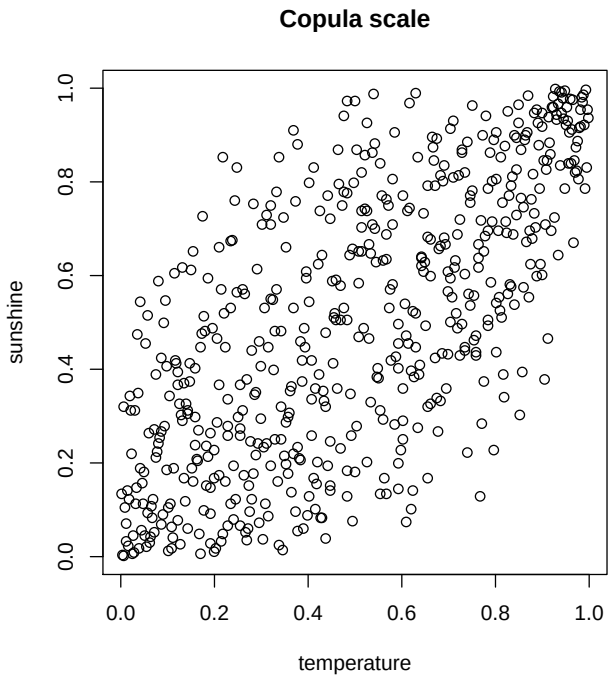


Margins:

🌡 F_1 is approximately normal

☀ F_2 is approximately beta distributed with support $[0, 15]$

Example: The “Biergarten” weather derivative



Copula:

- Might be modeled with the Gaussian copula C
- Hence, we have specified the joint law $F(x, y) = C(F_1(x), F_2(y))$

Agenda

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3) Building new models

A (simple) portfolio-default model

- Consider $d = 100$ loans with default times $(\tau_1, \ldots, \tau_d)$
- Assumptions: Unit notional, recovery rate 40%

- Choice of marginal laws: $F_i(x_i) = 1 - e^{-\lambda_i x_i}$ (exponential)
- Choice of copula: (Clayton)

$$C_\varphi(u_1, \ldots, u_d) = \varphi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d))$$
$$\varphi(x) = (1 + x)^{-1/\vartheta}, \quad \varphi^{-1}(x) = x^{-\vartheta} - 1, \quad \vartheta \in (0, \infty)$$

- Important: Clayton copula has lower tail dependence

3) Building new models

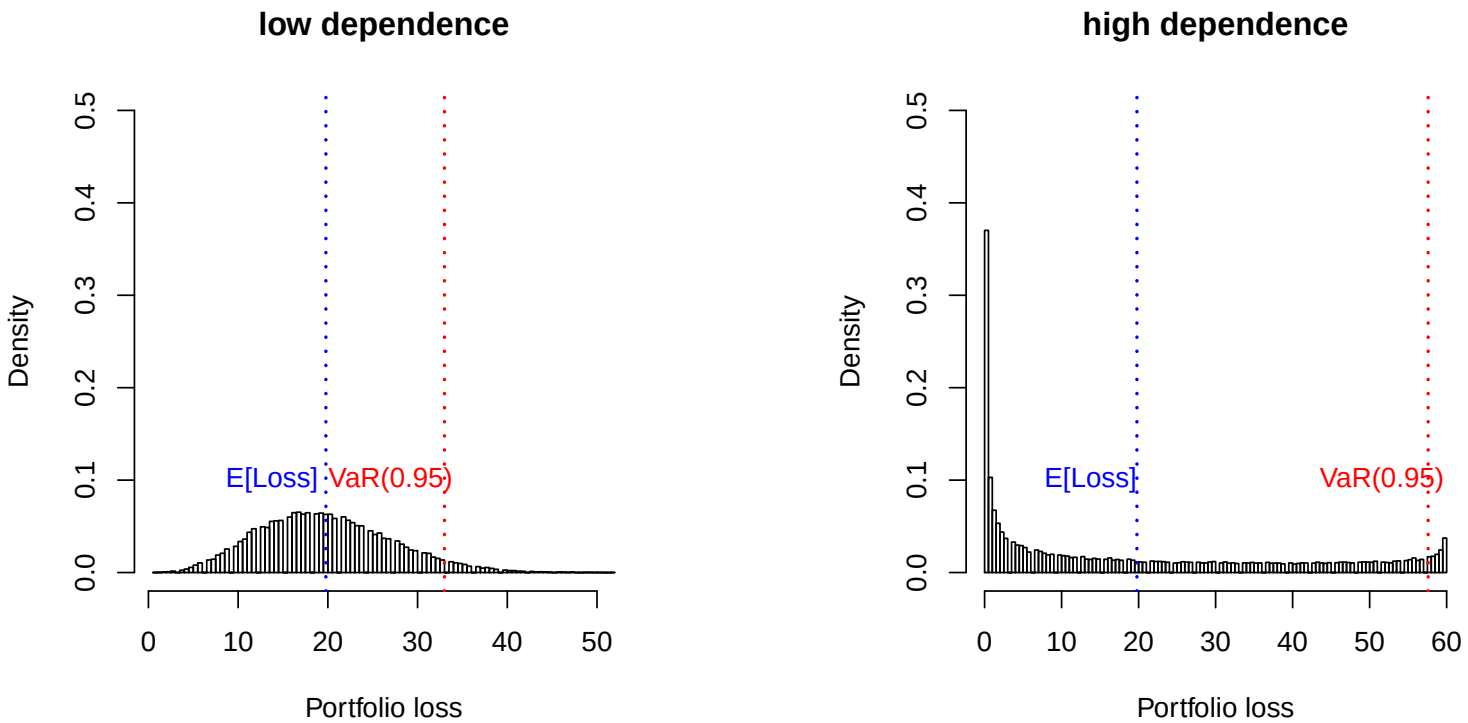
Algorithm (Simulating the portfolio loss)

1. Setup: Default probabilities F_i, maturity T, copula C, recovery rates, notionals
2. For each Monte Carlo run $j = 1, \ldots, m$
   - 2.1 Simulate from the copula $(U_1, \ldots, U_d) \sim C$
   - 2.2 Derive the default times via $\tau_i = F_i^{-1}(U_i)$, for $i = 1, \ldots, d$
   - 2.3 Compute the portfolio loss up to maturity T in run j denoted $L_T^{(j)}$
3. Estimate the portfolio loss distribution from $L_T^{(1)}, \ldots, L_T^{(m)}$

The portfolio loss distribution is required for:

- Risk management: VaR, sensitivities, ...
- Derivative pricing: CDOs, basket CDS, ...

3) Building new models



$d = 100$ credits, unit notional, $R = 40\%$, clayton copula, $\mathbb{P}(\tau_i \leq T) \approx 1/3$

Agenda

What can copulas do for you?

1) They **describe and measure dependence** between random variables
2) They make it possible to **identify dependence**
3) They allow us to **construct new multivariate distributions**
4) Current **research topics**

Current research
Pair-copula constructions

Idea

- Represent d-variate density f via pairs

$$f_{1234} = [\prod_{i=1}^{4} f_i] \cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot c_{23;1} \cdot c_{24;1} \cdot c_{34;12}$$

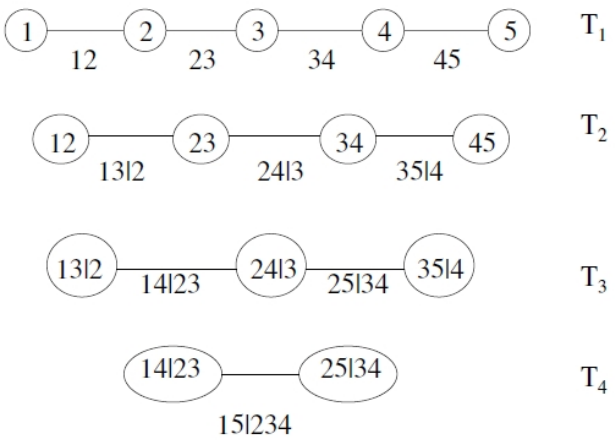


Figure 1 from Aas, Czado, Frigessi, Bakken (2009), *Pair-copula constructions of multiple dependence*

Tasks

- Design, estimation, simulation, ...

Useful resources

- Homepage: “Vine copula models”
  https://www.statistics.ma.tum.de/forschung/vine-copula-models/

- R-Package “VineCopula” by Erhardt et al.
  https://cran.r-project.org/web/packages/copula/copula.pdf

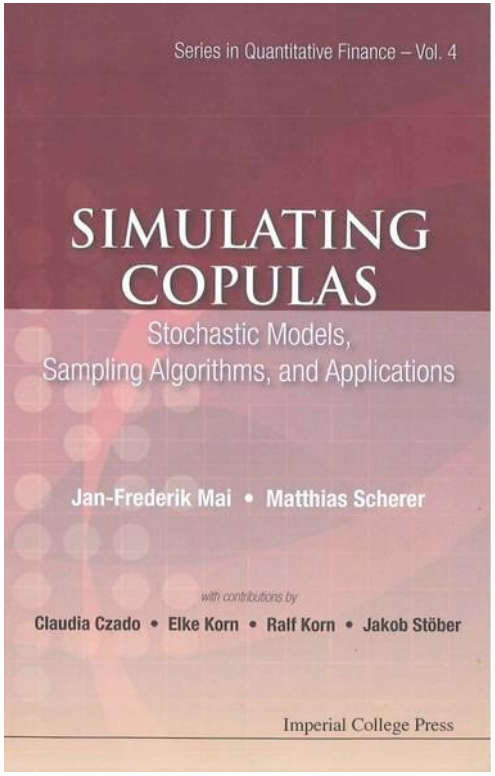
Current research
Simulation of copulas

Tasks

- Constructing stochastic models for copulas
- Efficient numerical implementations
- Challenge: large dimensions d

Useful resources

- R-Package “copula” by Hofert et al.
  https://cran.r-project.org/web/packages/copula/copula.pdf



Current research

VaR bounds / risk aggregation

Tasks

- Computing sharp lower (resp. upper) bounds on the distribution of a function (e.g. sum) of a number of dependent risks $L_1, \ldots, L_d$ with given marginal laws

$$\sup_{\text{copula } C} \Big\{ \mathrm{VaR}_\alpha(L_1 + \ldots + L_d) : F(\cdot) = C(F_1(\cdot), \ldots, F_d(\cdot)) \Big\}$$

rearranged matrix

	1	2	3	SUM	
1	4.47723	4.22233	3.20084	11.9004	WORST α-VaR
2	4.22233	4.47723	3.33013	12.0297	
3	3.47214	3.80384	4.77350	12.0495	
4	2.97360	4.00000	5.12372	12.0973	
5	5.54654	3.08248	3.47214	12.1012	
6	3.08248	3.47214	5.54654	12.1012	
7	4.77350	3.33013	4.00000	12.1036	
8	4.00000	3.62910	4.47723	12.1063	
9	6.07107	2.97360	3.08248	12.1271	
10	5.12372	3.20084	3.80384	12.1284	
11	3.20084	6.07107	2.87298	12.1449	
12	3.62910	5.54654	2.97360	12.1492	
13	3.80384	4.77350	3.62910	12.2064	
14	2.69274	2.77964	6.74597	12.2184	
15	6.74597	2.69274	2.77964	12.2184	
16	2.77964	6.74597	2.69274	12.2184	
17	2.87298	5.12372	4.22233	12.2190	
18	3.33013	2.87298	6.07107	12.2742	
19	7.66025	2.53553	2.61158	12.8074	
20	2.53553	2.61158	7.66025	12.8074	
21	2.61158	7.66025	2.53553	12.8074	
22	2.46410	9.00000	2.39683	13.8609	
23	9.00000	2.39683	2.46410	13.8609	
24	2.39683	2.46410	9.00000	13.8609	
25	2.33333	11.24745	2.27327	15.8541	
26	11.24745	2.27327	2.33333	15.8541	
27	2.27327	2.33333	11.24745	15.8541	
28	16.32051	2.21634	2.16228	20.6991	
29	2.16228	16.32051	2.21634	20.6991	
30	2.21634	2.16228	16.32051	20.6991	
SUM	136.01932	136.01932	136.01932		

Example by Giovanni Puccetti.

Useful resources

- "The Rearrangement Algorithm Project"
  https://sites.google.com/site/rearrangementalgorithm/

- R-Package "QRM" by Pfaff et al.
  https://cran.r-project.org/web/packages/QRM/index.html

Insurance T

Thursday, November 19, 2015

The times of IID are over

Scientists believe that the role of dependence has been underestimated for a long time in the insurance industry. An insurance company hardly runs into substantial trouble if individual insurance contracts are mis-priced. Events where many things go wrong at the same time are those that create true distress. The times of IID are over. Copulas enter the world of insurance and will play a pivotal role in the future.

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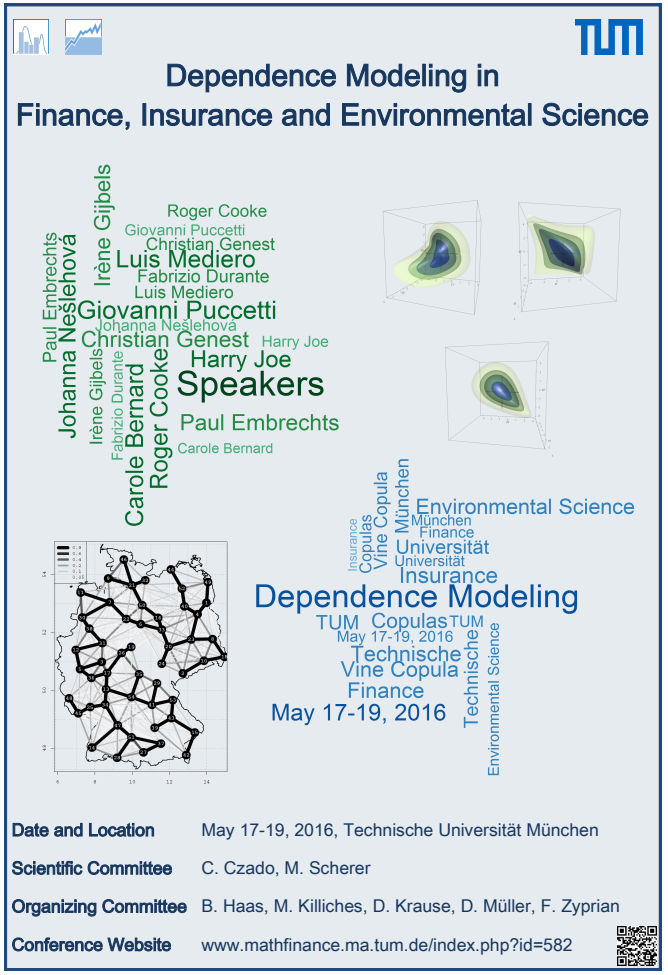
Thank you for your attention