Capital Calculation in Insurance

Mathieu Cambou

EdgeLab
École Polytechnique Fédérale de Lausanne (EPFL)

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Capital Calculation in Insurance

Economic	Regulatory
Liquid assets to reserve, as a cushion against future adverse events.	Protect policyholders against events that may affect solvency.
Internal methodology.	Supervised methodology.
Internal view of risk.	Industry-wise view of risk.

Research challenges

Challenge 1: Model uncertainty

The design of mathematical models, for deriving capital, raises questions of appropriatness at many levels (e.g. choice of the risk factors, choice of the models, ...).

$\hookrightarrow$ How should this uncertainty be quantified?

$\hookrightarrow$ How should this uncertainty be accounted?

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Model Uncertainty and Scenario Aggregation
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Challenge 2: Numerical estimates

Capital calculation can rarely be derived analytically.

$\hookrightarrow$ Numerical estimates are used, mainly Monte-Carlo estimates.

$\hookrightarrow$ Classical estimation error converges slowly, at best $n^{-1/2}$

B Mathieu Cambou, Marius Hofert, Christiane Lemieux
Quasi-Random Numbers for Copula Models
Statistics and Computing, 2017

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Replicating Portfolio Approach to Capital Calculation

- We provide a novel dynamic, tractable and path-dependent construction of the replicating portfolio,
- for monetary risk measures (VaR and ES),
- treat both the real-world and risk-neutral sampling measures.

Outline

Outline

Replicating Portolio Theory

Monte-Carlo Analysis

Examples

- Geometric Brownian Motion
- Put Option in Black–Scholes Model

Ingredients

- Fixed ultimate time horizon T
- $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with real-world measure $\mathbb{P}$
- All values and cashflows discounted by some numeraire
- Corresponding risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$

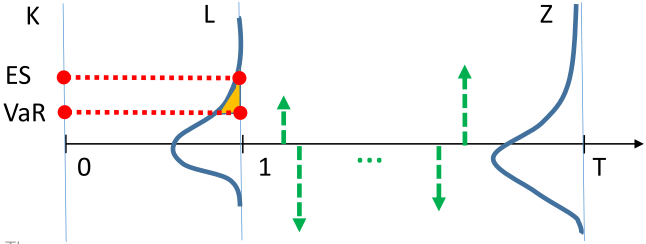
Capital Calculation in Stylized Form

- Terminal loss of asset-liability portfolio $Z \in L^2(\mathbb{Q})$ at T
- Portfolio is fairly price at $t = 0$ such that $\mathbb{E}^{\mathbb{Q}}[Z] = 0$
- One-year loss is given by $L = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_1]$

Goal: solvency capital calculation:

$$K = \rho[L]$$

where ρ is placeholder for either VaR_α or ES_α

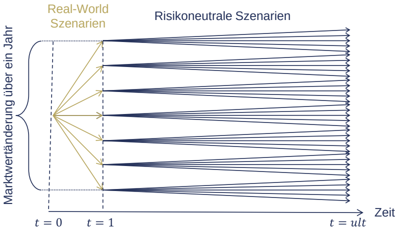


Problems

Simulating liability cash flows is costly:

- large time horizon $T \geq 40$ years
- path dependence:
  - embedded options, e.g. minimum rate guarantees
  - management & regulatory rules, e.g. policyholder participation
  - policyholder behaviour, e.g. lapsing

Nested simulation is computationally extremely costly:



Source: DAV 2015

Replicating Portfolio Approach

Goal: approximate Z in $L^2(\mathbb{Q})$, and thus L in $L^1(\mathbb{P})$, by a portfolio invested in m financial instruments

$$\mathbf{G}_t = (G_{1t}, \ldots, G_{mt})^\top$$

that can be efficiently simulated.

Chaos Expansion

This approximation is a computational problem. We can assume financial market model is complete: find strategy ψ_t such that

$$v + \int_0^T \psi_t \, d\mathbf{G}_t = Z$$

Idea: use "$\mathbf{G}_t$-chaos", for $m = 1$:

$$\psi_t = \phi_1(t) + \sum_{k=2}^{\infty} \int_{0<s_1<\cdots<s_{k-1}<t} \phi_k(s_1, \ldots, s_{k-1}, t) \, d\mathbf{G}_{s_1} \cdots d\mathbf{G}_{s_{k-1}}$$

where $\phi_1, \phi_2, \ldots$ are deterministic functions, and

$$\phi_k(s_1, \ldots, s_{k-1}, t)$$

is obtained by projecting Z on $d\mathbf{G}_{s_1} \cdots d\mathbf{G}_{s_{k-1}}$

Dynamic Portfolio Strategies for $m = 1$ Instrument

- Fix partition $0 = t_0 < t_1 < \cdots < t_N = T$ containing $t_j = 1$
- Write
$$\Delta G_j = G_{t_j} - G_{t_{j-1}}$$
- Chaos expansion: portfolio strategies are linear in the running product of gains ΔG_j
- $\mathcal{P}$ family of $\mathcal{J}$ where $\mathcal{J}$ is a subset of $\{1, \ldots, N\}$
- For any $\mathcal{J} \in \mathcal{P}$ define corresponding product of gains
$$\Delta \mathbf{G}_{\mathcal{J}} = \prod_{j \in \mathcal{J}} \Delta G_j$$
- Absence of arbitrage: $\mathbf{G}_t$ is a $\mathbb{Q}$-martingale:
$$\mathbb{E}^{\mathbb{Q}} \left[\Delta \mathbf{G}_{\mathcal{J}} \mid \mathcal{F}_{t_j} \right] = 0 \quad \text{for all } j < \min \mathcal{J}$$

Dynamic Portfolio Strategies for $m = 1$ Instrument

- Any choice of $\boldsymbol{\phi} = \{\phi_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{P}\} \in \mathbb{R}^{|\mathcal{P}|}$ and initial wealth v gives self-financing portfolio with value process

$$V_t^{v,\boldsymbol{\phi}} = v + \sum_{\mathcal{J} \in \mathcal{P} | t_{\max \mathcal{J}} \le t} \phi_{\mathcal{J}} \Delta \mathbf{G}_{\mathcal{J}}.$$

- Absence of arbitrage implies that $V_t^{v,\boldsymbol{\phi}}$ is a $\mathbb{Q}$-martingale.
- Positions in the instruments $\mathbf{G}_t$ path-dependent: $\bar{j} = \max \mathcal{J}$

$$\phi_{\mathcal{J}} \Delta \mathbf{G}_{\mathcal{J}} = \underbrace{\phi_{\mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{\bar{j}\}} \Delta G_j}_{\text{position}} \times \underbrace{\left(G_{t_{\bar{j}}} - G_{t_{\bar{j}-1}} \right)}_{\text{gain over } (t_{\bar{j}-1}, t_{\bar{j}}]}$$

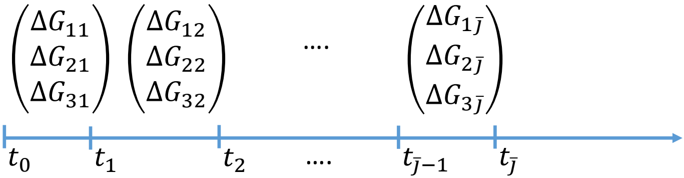
Example: first order portfolio for $m = 1$

- Assume $|\mathcal{J}| = \{j\}$ for all $\mathcal{J} \in \mathcal{P}$
- Obtain first order portfolio with value process

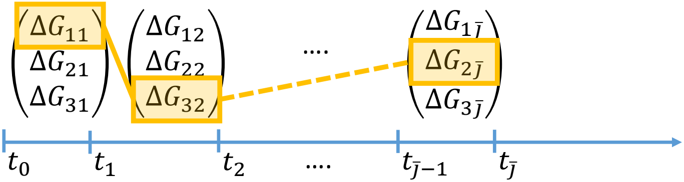
$$V_t^{v,\phi} = v + \sum_{t_j \leq t} \phi_j \Delta G_j$$

for the components $\phi_j = \phi_{\{j\}}$

Dynamic Portfolio Strategies for $m > 1$



Dynamic Portfolio Strategies for $m > 1$



Simplifying Notation (for $m = 1$)

- Portfolio gains up to one year

$$\mathbf{A} = \left(\Delta \mathbf{G}_{\mathcal{J}} \mid t_{\max \mathcal{J}} \leq 1\right)^{\top}$$

- Portfolio gains up beyond one year

$$\mathbf{B} = \left(\Delta \mathbf{G}_{\mathcal{J}} \mid t_{\max \mathcal{J}} > 1\right)^{\top}$$

- Portfolio values of $V_t^{v,\phi}$ at $t = 0, 1, T$ become

$$V_0^{v,\phi} = v, \quad V_1^{v,\phi} = v + \phi_A^{\top} \mathbf{A}, \quad V_T^{v,\phi} = v + \phi_A^{\top} \mathbf{A} + \phi_B^{\top} \mathbf{B}.$$

Replicating Portfolio

- Choose (v, ϕ) that solves the $L^2(\mathbb{Q})$-minimization problem

$$\min_{(v,\phi)\in\mathbb{R}^{1+|\mathcal{P}|}} \left\| Z - V_T^{v,\phi} \right\|_{L^2(\mathbb{Q})} . \tag{P}$$

- Corresponding $V_t^{v,\phi}$ is called replicating portfolio (RP)

Formal Solution

$\mathbb{Q}$-martingale property: $\mathbb{E}^{\mathbb{Q}}[\mathbf{A}] = 0$, $\mathbb{E}^{\mathbb{Q}}[\mathbf{B}] = 0$, and $\mathbb{E}^{\mathbb{Q}}[\mathbf{A}\mathbf{B}^\top] = 0$.

The formal solution of (P) is thus given by

$$v = 0, \quad \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} = \mathcal{N}^{-1} \mathbb{E}^{\mathbb{Q}} \left[\begin{pmatrix} \mathbf{A} Z \\ \mathbf{B} Z \end{pmatrix} \right]$$

with (block-)diagonal Gram matrix

$$\mathcal{N} = \mathbb{E}^{\mathbb{Q}} \left[\begin{pmatrix} \mathbf{A}\mathbf{A}^\top & 0 \\ 0 & \mathbf{B}\mathbf{B}^\top \end{pmatrix} \right].$$

Note: $\mathcal{N}$ may be close to singular due to possible strong correlation between the instruments $\mathbf{G}_t$.

- Numerical problems for their inverse.
- Closed form $\mathcal{N}$ preferred (e.g. polynomial models)

Capital Approximations

- Denote the residual from the $L^2(\mathbb{Q})$-projection:

$$\epsilon = Z - \phi_A^\top \mathbf{A} - \phi_B^\top \mathbf{B}$$

- One-year loss:

$$L = \phi_A^\top \mathbf{A} + \mathbb{E}^{\mathbb{Q}}\left[\epsilon \mid \mathcal{F}_1\right]$$

- Two approximations for L:

$$\begin{aligned} L_1 &= \phi_A^\top \mathbf{A} \\ L_2 &= \phi_A^\top \mathbf{A} + \epsilon = Z - \phi_B^\top \mathbf{B} \end{aligned}$$

- Two approximations for capital requirement K:

$$\begin{aligned} K_1 &= \rho[L_1] = \rho[\phi_A^\top \mathbf{A}] \\ K_2 &= \rho[L_2] = \rho[Z - \phi_B^\top \mathbf{B}] \end{aligned}$$

Industry Standard Static First Order RP

Static first order RP: buy and hold

Formal: $N = 2$, $\mathbf{A} = \mathbf{G}_1 - \mathbf{G}_0$, $\mathbf{B} = \mathbf{G}_T - \mathbf{G}_1$, $\phi_A = \phi_B = \psi$.

- $L^2(\mathbb{Q})$-minimization problem (P):

$$\min_{(v,\psi)\in\mathbb{R}^{1+m}} \left\| Z - v - \psi^\top(\mathbf{A}+\mathbf{B}) \right\|_{L^2(\mathbb{Q})}$$

- The formal solution is given by

$$\widetilde{\phi}_A = \widetilde{\mathcal{N}}^{-1}\mathbb{E}^{\mathbb{Q}}\left[(\mathbf{A}+\mathbf{B})Z\right], \quad \widetilde{\phi}_B = \widetilde{\phi}_A, \quad \widetilde{v} = 0$$

  with Gram matrix

$$\widetilde{\mathcal{N}} = \mathbb{E}^{\mathbb{Q}}\left[\, (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^\top \right].$$

Outline

Simulation-based $L^2(\mathbb{Q})$ Projection (P)

- Assume $\mathbf{A}Z, \mathbf{B}Z \in L^2(\mathbb{Q})$.
- Simulate n i.i.d. copies of $(\mathbf{A}, \mathbf{B}, Z)$ under $\mathbb{Q}$:

$$\left(\mathbf{A}^{(j)}, \mathbf{B}^{(j)}, Z^{(j)}\right), \quad j = 1 \dots n$$

- We obtain the unbiased estimators

$$\begin{pmatrix} \widehat{\phi_A} \\ \widehat{\phi_B} \end{pmatrix} = \mathcal{N}^{-1} \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix} \mathbf{A}^{(j)} Z^{(j)} \\ \mathbf{B}^{(j)} Z^{(j)} \end{pmatrix}$$

- LLN: $\left(\widehat{\phi_A}, \widehat{\phi_B}\right) \to (\phi_A, \phi_B)$ a.s. as $n \to \infty$.
- Central limit theorem ✓

Monte-Carlo Estimates of Capital Approximations

Estimators of the solvency capital approximation K_i:

$$\widehat{K_1} = \widehat{v} + \rho\left[\widehat{\phi_A}^\top \mathbf{A} \mid \mathcal{G}\right]$$
$$\widehat{K_2} = \rho\left[Z - \widehat{\phi_B}^\top \mathbf{B} \mid \mathcal{G}\right]$$

where $\mathcal{G}$ is σ-algebra generated by the sample $(\mathbf{A}^{(j)}, \mathbf{B}^{(j)}, Z^{(j)})$.

Theorem: Monte–Carlo estimates asymptotically consistent:

$$\widehat{K}_i \to K_i \text{ a.s. as } n \to \infty$$

Monte-Carlo Error

The total capital estimation error amounts to

$$\left\| K - \widehat{K}_i \right\|_{L^2(\mathbb{Q})} \leq \underbrace{|K - K_i|}_{\text{approximation error}} + \underbrace{\left\| K_i - \widehat{K}_i \right\|_{L^2(\mathbb{Q})}}_{\text{Monte-Carlo error}}$$

Theorem: For $\rho = \mathrm{ES}_\alpha$, asymptotically for large n:

$$\left\| K_i - \widehat{K}_i \right\|_{L^2(\mathbb{Q})} \leq \sqrt{\frac{1}{n}} \times \underbrace{\mathrm{MCE}_i}_{\text{constant}}$$

Numerical examples show: approximation and Monte-Carlo errors trade off ($\rightarrow$ balanced choice of $|\mathcal{P}|$ on a case by case basis)

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Replicating Portolio Theory

Monte-Carlo Analysis

Examples

- Geometric Brownian Motion
- Put Option in Black–Scholes Model

Sources of Static Incompleteness of the Insurance Market

Sources for incompleteness under static hedging with the underlying financial instruments (standard approach):

- insurance liability cash flows are nonlinear functions of the financial instruments
- insurance liability cash flows are path-dependent functions of the financial instruments

These effects superpose in practice. In the following (simple) examples, we illustrate these effects.

Parameters: $n = 5000$, number of MC runs $= 1000$ (for MC error)

Time partition: quarterly rebalancing: $t_j = j/4$

Outline

Geometric Brownian Motion

- Scalar $\mathbb{P}$-Brownian motion W_t
- Constant market price of risk $\gamma = 0.1$
- $m = 1$ financial instrument with gains process

$$G_t = W_t + \gamma t$$

- Define $\mathbb{Q}$-martingale, with loading $\lambda = -0.2$,

$$M_t = \exp\left(\lambda G_t - \frac{\lambda^2}{2} t\right)$$

  and assume

$$\text{one-year loss:} \quad L = M_1 - 1$$

$$\text{terminal loss:} \quad Z = M_T - 1$$

- Risk measure $\rho = \mathrm{ES}_{99\%}$
- Capital requirements and approximations normalised: $K = 1$

Wiener Chaos Expansion

- Risk-neutral projection measure $\mathbb{Q}$.
- Wiener chaos expansion theory: orthogonal series in $L^2(\mathbb{Q})$

$$M_t - 1 = \sum_{k=1}^{\infty} \int_{0<s_1<\cdots<s_k\leq t} \lambda^k \, dG_{s_1} dG_{s_2} \cdots dG_{s_k}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} t^{k/2} \underbrace{H_k\left(\frac{G_t}{\sqrt{t}}\right)}_{\text{Hermite poly}}.$$

- Comparing with

$$V_t^{v,\phi} = v + \sum_{\mathcal{J}\in\mathcal{P}|t_{\max \mathcal{J}}\leq t} \phi_{\mathcal{J}} \Delta \mathbf{G}_{\mathcal{J}} = v + \sum_{\mathcal{J}\in\mathcal{P}|t_{\max \mathcal{J}}\leq t} \phi_{\mathcal{J}} \prod_{j\in\mathcal{J}} \Delta G_j$$

suggests that $v = 0$ and

$$\phi_{\mathcal{J}} = \lambda^{|\mathcal{J}|}, \tag{3.1}$$

asymptotically for $N \to \infty$.

One-year Loss Approximations: Exact Formulas

We obtain, for $t = 1$,

$$L_1^{\mathbb{Q}} = \sum_{k=1}^{J} \frac{\lambda^k}{k!} H_k\left(G_1\right)$$

and

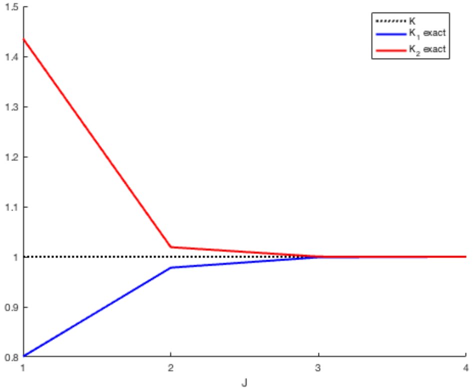
$$L_2^{\mathbb{Q}} = L_1^{\mathbb{Q}} + \epsilon^{\mathbb{Q}}$$

with

$$\epsilon^{\mathbb{Q}} = M_T - 1 - \sum_{k=1}^{J} \frac{\lambda^k}{k!} T^{k/2} H_k\left(\frac{G_T}{\sqrt{T}}\right)$$

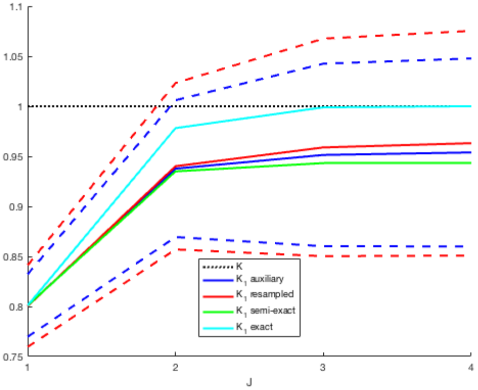
for varying degree of path-dependence $|\mathcal{J}| \leq J = 1, 2 \ldots$

Capital Approximations: Exact Approximation Errors



- Industry standard static approximations correspond to $J = 1$.
- Higher order RPs capture nonlinearities of liability cash flows significantly better, and for $J \geq 3$ extremely well.

Capital Approximations: MC based K_1



- Semi-exact (green) uses exact integrand (3.1).
- MC based approximations K_1 are lower biased due to time-discretisation of stochastic integral.
- MC errors (dashed) moderately increasing in J.

Loss & RP Trajectories

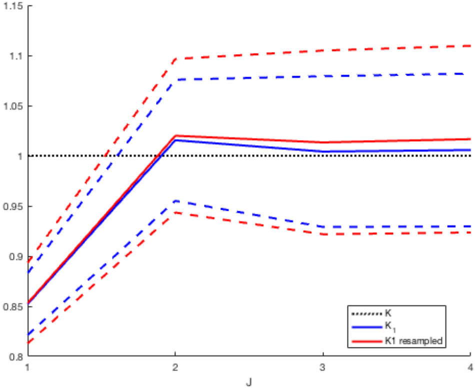
- Loss trajectory $L_t = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t]$ (blue) and quarterly rebalancing RP (red) in first year

Outline

Put Option in Black–Scholes Model

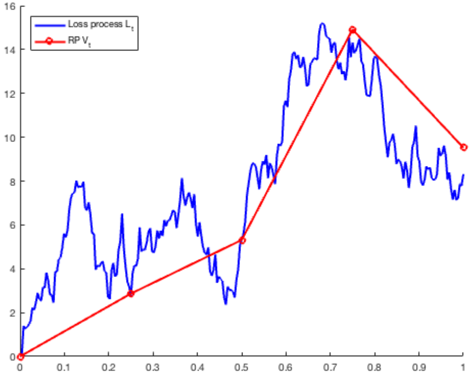
- Scalar $\mathbb{P}$-Brownian motion W_t
- Constant market price of risk $\gamma = 0.1$
- $m = 1$ financial instrument with volatility $\sigma = 0.2$ and gains process
$$G_t = 100 \exp\left(\sigma(W_t + \gamma t) - \frac{\sigma^2}{2} t\right)$$
- European put option with ATM strike $K = 100$ and maturity $T = 5$, denote by $P_t = \mathbb{E}^{\mathbb{Q}}[(K - G_T)^+ \mid \mathcal{F}_t]$ time-t price
- Assume
$$\begin{aligned} \text{one-year loss:} \quad & L = P_1 - P_0 \\ \text{terminal loss:} \quad & Z = (K - G_T)^+ - P_0 \end{aligned}$$
- Risk measure $\rho = \mathrm{ES}_{99\%}$
- Capital requirements and approximations normalised: $K = 1$

Capital Approximations: MC based K_1



- MC based approximations K_1 are accurate for $J \geq 2$
- MC errors (dashed) moderately increasing in J.

Loss & RP Trajectories



- Loss trajectory $L_t = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t]$ (blue) and quarterly rebalancing RP (red) in first year

Conclusion

- Dynamic path-dependent RP for capital calculation captures nonlinear path-dependence of liability cash flows very well.
- VaR and ES capital estimates asymptotically consistent under $\mathbb{P}$ and $\mathbb{Q}$ sampling if chaotic representation property holds.
- Numerical example illustrates that dynamic path-dependent RP outperforms industry standard static RP.
- Can be readily built into existing projection tools in practice.
- Ongoing: real-world study (jointly with German Insurance Company)

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