Long-Term Risks in Life Insurance and Pensions

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Veranstaltung des Fachkreises

Versicherungsmathematik am 22. November 2017 in Stuttgart

Motivation

- Fair value based accounting standards for insurers has enhanced the transparency of their balance sheets
  - Stressing the exposure of life insurers' balance sheets to a variety of financial and biometric factors
- Across all developed countries, life insurance and pension markets (particularly providers of long-term investment guarantees or lifelong benefits) have experienced (potential) distress due to
  - Low interest rates regimes
  - Increased expectation of life

$\rightarrow$ How shall individuals ensure retirement security in such an environment?

Long-term risks in life insurance and pensions

- Financial market risk and biometric risk
  - *Tonuity: A novel-individual oriented retirement plan*
    - joint with P. Hieber (University of Ulm), J. Klein (Allianz Life)
    - better longevity risk-sharing between the insurance companies and the policyholders, longevity risk margin
  - *The impact of longevity and investment risk on a portfolio of life insurance liabilities*
    - joint with A.R. Bacinello (University of Trieste), P. Millossovich (Cass Business School, City University London)
    - assessing the joint impact of demographic and financial risk on the market valuation of life insurance liabilities

Chen, Hieber and Klein (2017)

Tonuity: A novel-individual oriented retirement plan

Motivation I

How do we get sufficient **retirement income**?

- Desirable products (from policyholders' perspective)
  - not too costly
  - providing good protection against longevity risk
  - secure cash flows in advanced ages

Motivation II: retirement products

- Annuity
  - longevity protection ($\surd$)
  - **Solvency II**: Annuity products get more expensive (more risk capital needed).
- Tontine
  - Popular 17th century (FR, GB), today **"Le Conservateur"** (FR)
  - not good longevity protection
  - low risk capital required ($\surd$)

$\Rightarrow$ Ton~~tine/ann~~uity = Tonuity: better risk-sharing between the insurance companies and the policyholders

Annuity and Tontine: Payoff

Single premium P_0 at time $t = 0$.

Annuity: payoff $c(t)$ ($t \geq 0$) until death (residual life time $\zeta > 0$):

$$b^{[0]}(t) := \mathbf{1}_{\{\zeta > t\}}\, c(t)\,.$$

Tontine: **homogeneous cohort** of size n receives payoff $nd(t)$ ($t \geq 0$). Each **tontine** holder receives:

$$b^{[\infty]}(t) := \begin{cases} \mathbf{1}_{\{\zeta > t\}} \frac{nd(t)}{N_t} & \text{if } N_t > 0, \\ 0, & \text{else} \end{cases}\,.$$

where N_t is the number of surviving policyholders at time t.

Tontine: example

1st year

$d(1) = 800, N_1 = 8$

$nd(1)/N_1 = 800$

2nd year

$d(2) = 800, N_2 = 7$

$nd(2)/N_2 = 914.29$

3rd year

$d(3) = 720, N_3 = 7$

$nd(3)/N_3 = 822.86$

Mortality/longevity risk

We distinguish:

- Unsystematic (or hedgeble, or diversifiable...) mortality risk:
  - individual's lifetime is uncertain
  - can be diversified by pool size
- Systematic (or unhedgeable or aggregate) mortality risk:
  - the true underlying mortality law cannot be determined with certainty (e.g. unexpected medical progress or change in life style)
  - cannot be diversified by pool size

A simple mortality model

We follow, e.g., Lin, Cox [2005].

(1) Get survival probabilities ${}_tp_x$, $t \geq 0$ from past data (best-estimate survival probability)

(2) Draw a mortality shock ϵ, true survival probabilities are $({}_tp_x)^{1-\epsilon}$. (**systematic mortality risk**)

- ϵ is a r.v. with density $f_\epsilon(\varphi)$ and support on $(-\infty, 1)$

(3) Conditional on $\epsilon = \varphi$, the number of survivors is binomially distributed, i.e. $N(t) \sim \text{Bin}(n, {}_tp_x^{1-\varphi})$, (**unsystematic mortality risk**)

Policyholder utility

- Policyholder follows constant relative risk aversion (**CRRA**) utility

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

with risk aversion $\gamma \in [0, \infty) \setminus \{1\}$.

- Assumption: a rational retiree **without bequest motives** would choose $b(t)$ to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\eta t} u\big(b(t)\big) \,\mathrm{d}t \right],$$

with personal discount factor η, given an **actuarially fair** premium.

Theorem (Optimal payout function: Annuity and Tontine)

(a) *For an annuity product, we obtain*

$$c^*(t) = e^{\frac{1}{\gamma}(r-\eta)t} \cdot P_0 \cdot \left(\int_0^\infty e^{\left(\frac{r-\eta}{\gamma} - r\right)t} {}_t\bar{p}_x \, \mathrm{d}t \right)^{-1},$$

where ${}_t\bar{p}_x := \mathbb{E}[{}_tp_x{}^{1-\epsilon}]$. *(e.g. Yaari [1965])*

(b) *For a tontine product, we obtain*

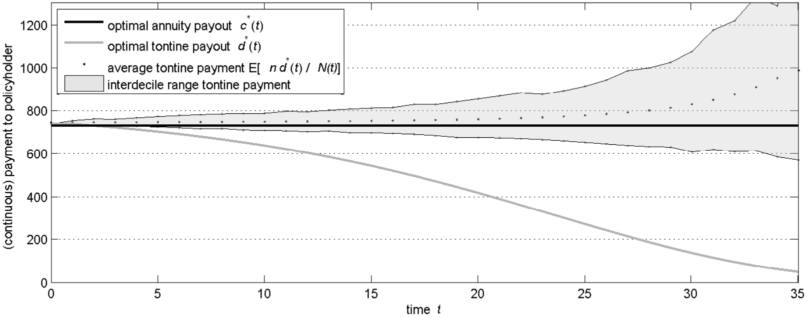
$$d^*(t) = \frac{e^{\frac{1}{\gamma}(r-\eta)t} \cdot P_0}{\left(\lambda^*\right)^{\frac{1}{\gamma}}} \cdot \frac{\kappa_{n,\gamma,\epsilon}({}_tp_x)}{\mathbb{E}\left[1 - \left(1 - {}_tp_x{}^{1-\epsilon}\right)^n\right]^{\frac{1}{\gamma}}},$$

with suitable $\kappa_{n,\gamma,\epsilon}$ *and* λ^*. *(e.g. Milevsky, Salisbury [2015])*

Numerical example: parameter choices

net premium $P_0 = P_0^{[0]} = P_0^{[\infty]} = 10\,000$	pool size $n = 100$	risk aversion $\gamma = 6$
risk-free rate $r = 4\%$	subjective discount rate $\eta = 4\%$	cost of capital rate $CoC = 6\%$
initial age $x = 65$	Gompertz-law $m = 88.721$, $b = 10$	mortality shock $\epsilon \sim \mathcal{N}_{(-\infty,1)}(\mu, \sigma^2)$ $\mu = -0.0035$, $\sigma = 0.0814$

Numerical example



Optimal payouts $c^*(t)$ and $d^*(t)$. Distribution $n \cdot d^*(t)/N(t)$.

Risk capital charge: Risk margin according to Solvency II

product		risk capital charge F_0
tontine	$n = 10$	63.68
	$n = 100$	5.40
	$n = 1\,000$	0.35
annuity		459.22

Risk capital charges

$$F_0 = \text{CoC} \cdot \sum_{t=0}^{\infty} e^{-r(t+1)} \cdot SCR(t)$$

for different pool sizes n.

Drawbacks Tontine/Annuity

Both products have advantages / disadvantages, mainly:

- For an annuity, the insurance company takes the **aggregate mortality risk**. This increases the cost of risk capital provision (a tontine does not).
- A tontine leads to a **volatile payoff at old ages** (an annuity does not).

Combine both products (**Ton~~tine/Ann~~uity = Tonuity**)?

Tonuity: Payoff

Idea: Switch between tontine and annuity payoff:

$$b_{[\tau]}(t) := \mathbb{1}_{\{0 \leq t < \min\{\tau,\zeta\}\}} \frac{n d_{[\tau]}(t)}{N(t)} + \mathbb{1}_{\{\tau \leq t < \zeta\}} c_{[\tau]}(t) ,$$

with **switching time** τ:

- A tonuity with switching time $\tau = 0$ is an **annuity**
- A tonuity with switching time $\tau \to \infty$ is a **tontine**
- **Volatile tontine payoff at old ages** is **replaced by** a secure **annuity** payoff

Theorem (Optimal payout function: Tonuity)

For a tonuity with switching time $\tau \in [0, \infty)$, we obtain

$$d^*_{[\tau]}(t)\Big|_{0\le t\le\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}\cdot P_0}{(\lambda^*)^{\frac{1}{\gamma}}}\cdot\frac{\kappa_{n,\gamma,\epsilon}({}_tp_x)}{\mathbb{E}\left[1-\left(1-{}_tp_x{}^{1-\epsilon}\right)^n\right]^{\frac{1}{\gamma}}},$$

$$c^*_{[\tau]}(t)\Big|_{t>\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}\cdot P_0}{(\lambda^*)^{\frac{1}{\gamma}}},$$

with suitable λ^* and $\kappa_{n,\gamma,\epsilon}$.

Proof: Chen, Hieber, Klein [2017].

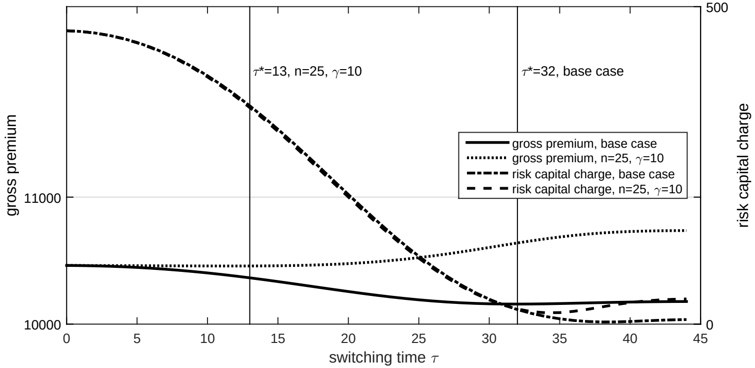
Product comparison

- It is necessary to find a way to weigh
  - life time benefit vs. the cost of the products
  - longevity risk aversion vs. the amount of risk capital charges F_0
- **Idea: Look at products with the same lifetime benefit.**
  - Buying $CEQ_{[\tau]}$ tonuities gives **lifetime benefit**

$$\mathbb{E}\left[\int_0^\infty e^{-\eta t} u\Big(CEQ_{[\tau]} \cdot b_{[\tau]}(t)\Big)\,\mathrm{d}t\right] = \mathbf{CEQ}_{[\tau]}^{\mathbf{1}-\gamma} \cdot \mathbf{U}^{[\tau]} = \mathbf{U}^{[\mathbf{0}]}\,.$$

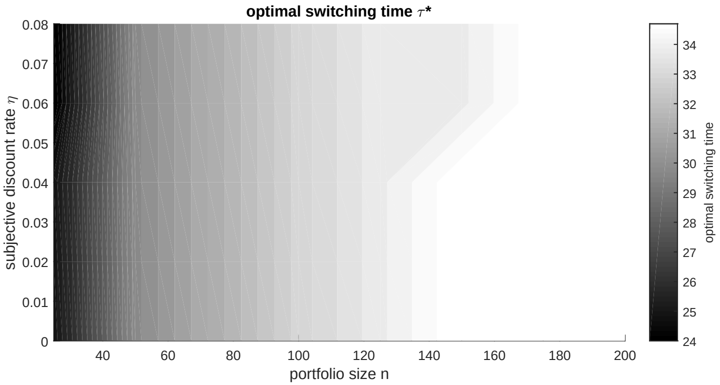
  - Choose $CEQ_{[\tau]}$ such that $\mathbf{CEQ}_{[\tau]}^{\mathbf{1}-\gamma} \cdot \mathbf{U}^{[\tau]}$ **is the same for any switching times** τ; Find best τ leading to lowest contribution $CEQ_{[\tau]} \cdot (P_0 + F_0[\tau])$.

Numerical analysis: Optimal switching time I



Gross premia $\mathrm{CEQ}_{[\tau]} \cdot (P_0 + F_0^{[\tau]})$ using the base-case parameter set ($\gamma = 6, n = 100$) and the case ($\gamma = 10, n = 25$)

Numerical analysis: Optimal switching time II



Discussion

- **Tonuities combine beneficial features** of annuities, tontines:
  - **Reduced solvency capital** provision (tontine).
  - **Secure income at old ages** (annuity).
- **Choice** between **tonuity** / **annuity** / **tontine** depends (mostly) on **longevity risk aversion**, **pool size**, **cost-of-capital rate**.

Bacinello, Chen, and Millossovich (2017)

The impact of longevity and investment risk on a portfolio of life insurance liabilities

Main message of Bacinello, Chen and Millossovich (2017)

- Analyze the effect of demographic and investment risk on the market valuation of life insurance liabilities for a portfolio of homogenous policyholders
- Systematic mortality risk overshadows process risk even for small portfolios
  - recognizing longevity risk as one of the most challenging factors affecting the life insurance business
- Under low interest rate levels, yet not even close to those currently experienced, the cost of guarantees offered may be hardly sustainable

Model setup

- A life insurance company which issues participating policies.
- At the initial time $t = 0$, the assets of the company are financed by two groups of stakeholders: policyholders, equity holder
- The balance-sheet at time 0 of the insurance company

Assets	Liabilities
W_0	$E_0 = (1 - \alpha)W_0$ $L_0 = \alpha W_0$
W_0	W_0

  - W_0: initial assets, E_0 total equity holders' initial contribution
  - L_0: overall premium equally contributed by N_0 homogeneous policyholders

Contract structure

- At maturity T, the total outstanding liability the insurance company

$$L_T = \begin{cases} \Psi_T & \text{if } N_T > 0 \\ 0 & \text{if } N_T = 0 \end{cases} = \Psi_T 1_{\{N_T > 0\}},$$

  where $1_{\mathcal{E}}$ is the indicator of the event $\mathcal{E}$.

- If $N_T > 0$, the liability depends on the assets value $W_T = W_0 e^R$ and the global payoff G_T guaranteed to surviving policyholders ($G_T = N_T B_T$)

$$\Psi_T = \begin{cases} W_T & \text{if } W_T < G_T \\ G_T & \text{if } G_T \le W_T \le \frac{G_T}{\alpha} \\ G_T + \delta(\alpha W_T - G_T) & \text{if } \frac{G_T}{\alpha} < W_T \end{cases}$$

  or $= G_T + \delta\alpha \left[W_T - \frac{G_T}{\alpha} \right]^+ - [G_T - W_T]^+$

Specification of B_T

- The individual guaranteed benefit B_T:

  (a) $B_T = b$: pure endowments, where the guarantee is fixed and the individual benefit B is therefore deterministic $B = b$;

  (b) $B_T = \rho\, a_T$: deferred whole life annuities guaranteeing each survivor the continuous payment ρ per year, starting at time T;

  (c) $B_T = b + [b\, \rho^g\, a_T - b]^+ = b \max\{1, \rho^g\, a_T\}$: pure endowments with attached a guaranteed annuity option.

Individual liability

- The individual liability at maturity T attributed to policyholder i is defined by

$$\ell_T^i = \frac{L_T}{N_T} 1_{\{\tau^i > T\}} = \frac{\Psi_T}{N_T} 1_{\{\tau^i > T\}} = \psi_T 1_{\{\tau^i > T\}}, \quad i = 1, \ldots, N_0, \quad (1)$$

  where τ^i denotes her residual lifetime. In particular, the liability attributed to each policyholder surviving at time T is then, on the set $\{N_T > 0\}$, equal to

$$\psi_T = B_T + \delta\alpha \left[w_T - \frac{B_T}{\alpha} \right]^+ - [B_T - w_T]^+ ,$$

  with $w_T = \frac{W_T}{N_T}$.
- Adding up the individual liabilities recovers the total liability: $L_T = \sum_{i=1}^{N_0} \ell_T^i$.

Modelling demographic risk

- The possibility of deviations between actual and expected mortality/survival rates:
  - the unsystematic risk, that can be diversified away through pooling
  - the systematic risk, that hits all policies in the same direction
    - In our case of survival benefits, it can be identified in the longevity risk, that is the risk of an overall decline in mortality rates
    - When it is present, even with a large portfolio there is a residual part of risk that cannot be eliminated

Stochastic framework

- We fix a probability measure Q which accounts for both unsystematic and systematic risk and, in particular, can depend on the portfolio size N_0
- Conditionally on a positive random variable Δ, the residual lifetimes $\tau^i, i = 1, \ldots, N_0$, are independent, and

$$Q(\tau^1 > t_1, \ldots, \tau^{N_0} > t_{N_0} | \Delta) = \prod_{i=1}^{N_0} Q(\tau^i > t_i | \Delta) = \prod_{i=1}^{N_0} \mathrm{e}^{-\Delta \int_0^{t_i} m(v) \mathrm{d}v}$$

  for any $t_i \geq 0$, $i = 1, \ldots, N_0$, where m is a deterministic force of mortality.
- Conditionally on Δ , the residual lifetimes are first jump times of independent inhomogeneous Poisson processes with common stochastic intensity $\mu_t = \Delta m(t)$.

Large portfolios

- We assume now there are infinitely many policyholders in the group
  - The risk neutral measure Q contains an adjustment for systematic risk only, as the portfolio size is large and all unsystematic risk has been diversified away
  - N_T gives now the number of survivors at time T among the subportfolio of policyholders with index $1, 2, \cdots N_0$
  - $\frac{N_T}{N_0} \to e^{-\Delta \int_0^T m(v)dv} := \pi^\Delta$ as $N_0 \to \infty$ almost surely under Q.

Large portfolio: individual outstanding liability

- The individual liability for the policyholder in the large portfolio case is given, on the set $\{\tau^i > T\}$

$$\ell_T^i(\infty) = \lim_{N_0 \to \infty} \ell_T^i = B_T + \delta\alpha(\infty)\left[\frac{w_0(\infty)e^R}{\pi^\Delta} - \frac{B_T}{\alpha(\infty)}\right]^+ - \left[B_T - \frac{w_0(\infty)e^R}{\pi^\Delta}\right]^+$$

  - We assume $\frac{W_0}{N_0} = w_0 \to w_0(\infty)$ positive and finite
  - $\frac{L_0}{N_0} = l_0 \to l_0(\infty) \leq w_0(\infty)$
  - $\alpha(\infty) := l_0(\infty)/w_0(\infty) \leq 1$ represents the leverage ratio for an insurer supporting a large portfolio
  - R is the log-return on the assets over the period $[0, T]$

Assumption about financial market

- We disregard stochasticity of the interest rates and assume that the market instantaneous short rate is a constant r
- The financial uncertainty comes only from the assets randomness
- We do not make specific assumptions on the dynamics of the assets value W, and just require independence between it and all demographic related variables

Valuation: finite portfolio

- The initial market value V_0^ℓ of the individual liability

$$
\begin{aligned}
V_0^\ell =& E\left[\mathrm{e}^{-rT}\ell_T^i\right] = E[\mathrm{e}^{-rT}B_T 1_{\{\tau^i>T\}}] + \delta\alpha E\left[\mathrm{e}^{-rT}\left[w_T - \frac{B_T}{\alpha}\right]^+ 1_{\{\tau^i>T\}}\right] \\
&- E\left[\mathrm{e}^{-rT}\left[B_T - w_T\right]^+ 1_{\{\tau^i>T\}}\right] \\
=& V_0^{\mathrm{g}} + \delta\alpha V_0^{\mathrm{b}} - V_0^{\mathrm{d}}, \qquad i = 1, \ldots, N_0.
\end{aligned}
$$

  - V_0^{g}, V_0^{b} and V_0^{d}, correspond to the values of the guaranteed amount, bonus option and default option
  - The value of the total liability is

$$
V_0^L = E\left[\mathrm{e}^{-rT}L_T\right] = E\left[\mathrm{e}^{-rT}\sum_{i=1}^{N_0}\ell_T^i\right] = N_0 V_0^\ell,
$$

Fair contract analysis

- A contract is fair for the policyholders if the initial market value of the outstanding liabilities equates their initial investment.

$$V_0^L = \alpha W_0 \quad \text{or, equivalently,} \quad V_0^\ell = \alpha w_0,$$

- It is particularly relevant to analyse the trade-off between contract parameters that implicitly define a fair policy.
  - participation coefficient δ associated with a fair contract:

$$\delta = \frac{\alpha w_0 - V_0^{\mathrm{g}} + V_0^{\mathrm{d}}}{\alpha V_0^{\mathrm{b}}},$$

Infinite portfolio

- In an infinite portfolio, the market value of the individual liability is given by

$$\begin{aligned} V_0^{\ell}(\infty) =& E[\mathrm{e}^{-rT}\ell^i(\infty)] \\ =& E[\mathrm{e}^{-rT}B1_{\{\tau^i>T\}}] \\ &+\delta\alpha(\infty)E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}-\frac{B}{\alpha(\infty)}\right]^+ 1_{\{\tau^i>T\}}\right] \\ &-E\left[\mathrm{e}^{-rT}\left[B-\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+ 1_{\{\tau^i>T\}}\right] \\ =& V_0^{\mathrm{g}}(\infty)+\delta\alpha(\infty)V_0^{\mathrm{b}}(\infty)-V_0^{\mathrm{d}}(\infty). \end{aligned}$$

Infinite portfolio: fair contract analysis

- Fairness of a contract in an infinitely large portfolio can only be defined at individual level.
- Fair contracts are then those for which

$$V_0^{\ell}(\infty) = \alpha(\infty) w_0(\infty).$$

- The fair participation coefficient

$$\delta(\infty) = \frac{\alpha(\infty) w_0(\infty) - V_0^{\mathrm{g}}(\infty) + V_0^{\mathrm{d}}(\infty)}{\alpha(\infty) V_0^{\mathrm{b}}(\infty)}.$$

Numerical results: parameter choices

- We now focus on the large portfolio case assumptions
- m Gompertz law of mortality, fitted to the survival probabilities ${}_tp^*_{40}$ implied by the projected life table IPS55 currently used in the Italian annuities market

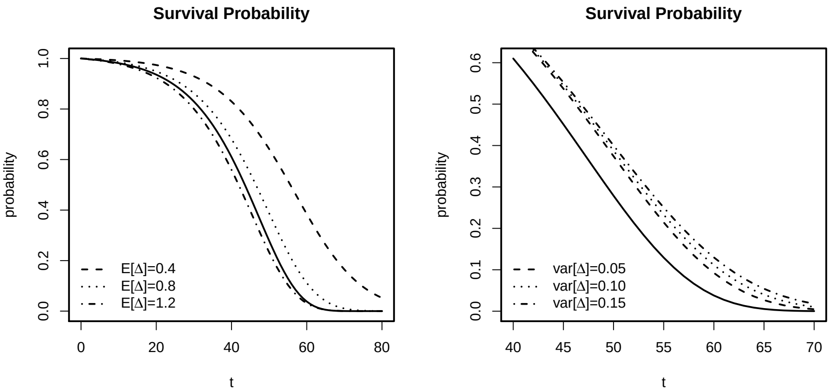
$$m(t) = \lambda c^{x+t}$$

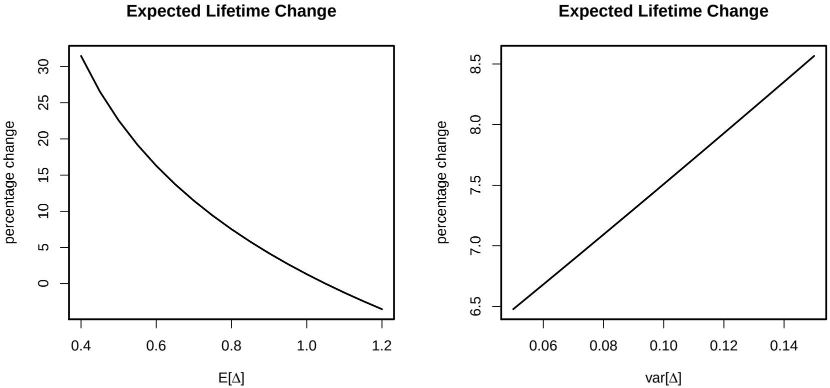
with $x = 40, \lambda = 2.6743 \cdot 10^{-5}, c = 1.098$.

- Gamma distributed with $Var[\Delta] = 0.1$ and the following scenarios:
  - $E[\Delta] = 0.4$ extreme longevity improvement scenario
  - $E[\Delta] = 0.8$ moderate longevity improvement scenario
  - $E[\Delta] = 1.2$ slight mortality worsening scenario
- instantaneous assets return normally distributed with mean r and σ

..Other parameters

- maturity $T = 25$;
- initial individual assets per contract $w_0 = 100$;
- initial contribution ratio $\alpha = 0.7$;
- riskless rate $r = 0.03$;
- assets volatility $\sigma = 0.15$;
- in cases (a) and (c), individual survival benefit $b = 150$;
- in case (b), instantaneous annuity amount $\rho = 10$;
- in case (c), guaranteed conversion rate $a^g \doteq 1/\rho^g = 15$.


Survival Probability
probability
t
E[Δ]=0.4
E[Δ]=0.8
E[Δ]=1.2
Survival Probability
probability
t
var[Δ]=0.05
var[Δ]=0.10
var[Δ]=0.15



Percentage change in $E[\tau^i]$ with respect to $E[\tau^*] = 41.73$ as $E[\Delta]$ and $var[\Delta]$ varies. The baseline case includes $E[\Delta] = 0.8$ and $var[\Delta] = 0.1$

Fair participation rate for case (b): effect of systematic mortality and guaranteed payment

	$E[\Delta] = 0.4$				$E[\Delta] = 0.8$				$E[\Delta] = 1.2$			
ρ	$\delta\%$	V_0^g	V_0^b	V_0^d	$\delta\%$	V_0^g	V_0^b	V_0^d	$\delta\%$	V_0^g	V_0^b	V_0^d
5.0	90.28	43	48	3	95.69	33	57	1	97.65	28	63	1
7.5	69.16	65	33	11	85.11	50	42	5	91.33	42	49	3
10.0	32.76	87	23	22	66.14	67	31	11	79.64	56	38	7
12.5	—	108	17	36	37.33	84	24	20	61.63	70	30	13
15.0	—	130	12	52	—	100	18	30	36.41	84	24	20

Case (b) for a large portfolio, different annuity rates ρ and values of $E[\Delta]$.

$$V_0^g(\infty) = E[\mathrm{e}^{-rT}\rho a_T 1_{\{\tau^i > T\}}];$$

$$V_0^b(\infty) = E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta} - \frac{\rho a_T}{\alpha}\right]^+ 1_{\{\tau^i > T\}}\right]$$

$$V_0^d(\infty) = E\left[\mathrm{e}^{-rT}\left[\rho a_T - \frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+ 1_{\{\tau^i > T\}}\right]$$

Fair participation rate for case (b): combined effect of systematic mortality and low interest rate

	$E[\Delta]=0.4$				$E[\Delta]=0.8$				$E[\Delta]=1.2$			
$r\%$	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}
1	—	196	6	108	—	140	10	59	—	113	15	39
2	—	129	13	52	8.17	97	19	28	45.47	79	25	17
3	32.76	87	23	22	66.14	67	31	11	79.64	56	38	7
4	76.54	59	36	8	87.93	47	45	4	92.75	40	51	2
5	92.21	40	50	3	95.96	33	57	1	97.60	28	62	1

Case (b) for a large portfolio, different risk free rates r and values of $E[\Delta]$

$$V_0^{\mathrm{g}}(\infty) = E[\mathrm{e}^{-rT}\rho a_T 1_{\{\tau^i>T\}}];$$

$$V_0^{\mathrm{b}}(\infty) = E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^{R}}{\pi^{\Delta}} - \frac{\rho a_T}{\alpha}\right]^{+} 1_{\{\tau^i>T\}}\right]$$

$$V_0^{d}(\infty) = E\left[\mathrm{e}^{-rT}\left[\rho a_T - \frac{w_0(\infty)\mathrm{e}^{R}}{\pi^{\Delta}}\right]^{+} 1_{\{\tau^i>T\}}\right]$$

Fair participation rate for case (b): combined effect of systematic mortality and asset volatility

	$E[\Delta] = 0.4$			$E[\Delta] = 0.8$			$E[\Delta] = 1.2$		
σ	$\delta\%$	V_0^b	V_0^d	$\delta\%$	V_0^b	V_0^d	$\delta\%$	V_0^b	V_0^d
0.100	—	14	14	53.37	22	5	78.76	30	2
0.125	9.36	18	18	60.23	27	8	78.58	34	5
0.150	32.76	23	22	66.14	31	11	79.64	38	7
0.175	48.01	28	26	71.08	36	15	81.21	42	10
0.200	58.61	33	30	75.19	40	18	82.93	46	12

Case (b) for a large portfolio, different volatilities σ and values of $E[\Delta]$. The values of the guaranteed amount are $V_0^g = 87$ for $E[\Delta] = 0.4$, $V_0^g = 67$ for $E[\Delta] = 0.8$, $V_0^g = 56$ for $E[\Delta] = 1.2$.

$$
\begin{aligned}
V_0^g(\infty) =& E[\mathrm{e}^{-rT} \rho a_T 1_{\{\tau^i > T\}}]; \\
V_0^b(\infty) =& E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta} - \frac{\rho a_T}{\alpha}\right]^+ 1_{\{\tau^i > T\}}\right] \\
V_0^d(\infty) =& E\left[\mathrm{e}^{-rT}\left[\rho a_T - \frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+ 1_{\{\tau^i > T\}}\right]
\end{aligned}
$$

Fair participation rate: effect of portfolio size

	$E[\Delta] = 0.4$			$E[\Delta] = 0.8$			$E[\Delta] = 1.2$		
N_0	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$
1	75.20	44.04	42.68	91.91	89.58	86.92	—	—	—
2	65.49	35.21	33.86	71.18	68.82	65.87	76.81	83.27	76.49
5	64.56	33.56	32.13	69.00	66.58	63.46	72.84	79.77	72.50
10	64.43	33.16	31.70	68.80	66.37	63.18	72.63	79.71	72.28
100	64.30	32.80	31.32	68.61	66.17	62.92	72.42	79.65	72.07
∞	64.29	32.76	31.28	68.59	66.14	62.89	72.40	79.64	72.04

Cases (a), (b) and (c) for a finite portfolio, different portfolio sizes and values of $E[\Delta]$. $\delta^{(a)}$, $\delta^{(b)}$ and $\delta^{(c)}$ are the fair participation rates for cases (a), (b) and (c).

Fair participation rates δ: Pricing measure adjusted to the portfolio size

N_0	$\phi(N_0)\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$
1	50	75.20	44.04	42.68
2	67	67.40	50.89	49.32
5	83	67.59	59.30	57.08
10	91	68.05	62.64	60.00
100	99	68.53	65.79	62.60
∞	100	68.59	66.14	62.89

Fair participation rates δ for cases (a), (b) and (c) with different portfolio sizes and size-adjusted risk neutral measures, $E^{(N_0)}[\Delta^{(N_0)}] = E^{(\infty)}[\Delta^{(\infty)}]\phi(N_0)$, where $\phi(N_0) = \frac{N_0}{N_0+1}$ and $E^{(\infty)}[\Delta^{(\infty)}] = 0.8$.

Concluding remark

- This paper aims at shedding some light on the interplay between demographic and investment risk affecting most life insurance products
- Systematic mortality risk overshadows process risk even for small portfolios
- Under low interest rate levels, yet not even close to those currently experienced, the cost of guarantees offered may be hardly sustainable

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